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THERMODYNAMICS, PART II: WORK, HEAT, AND TEMPERATURE CONCEPTS, AND AN EXAMINATION OF THE TEMPERATURE SCALE.

BY

J. L. FINCK, Ph.D.[1]

ABSTRACT.

In this paper the concepts of work, heat, and temperature are examined from the standpoint of a complete equation (1) [2] for the internal energy of a system. In an incomplete system energy necessarily remains with the system, which results in the inefficiency of a process. This inefficiency is, of course, appreciable for processes taking place at a finite speed, but is not necessarily eliminated by means of a quasistatic process.

The Kelvin temperature scale is shown to be based on an incomplete equation for the internal energy and, therefore, offers certain difficulties. A temperature scale based on a complete equation is shown to have advantages over the Kelvin scale, and it is further shown that the temperature of absolute zero as based on a complete equation is lower than that on the Kelvin scale.

A suggestion is offered whereby an actual system may be treated as complete and a temperature scale may be based on this.

1. INTRODUCTION.

The present considerations are based directly on the results developed by the writer in a previous paper (1). For this reason it may be well, at the start, to state briefly the conclusions which were arrived at in this paper, which are as follows:

(*a*) The *number* of independent variables which are selected to define a given thermodynamic system is crucial to the concepts of "equilibrium" and "reversibility," as well as to the second law of thermodynamics. If this number of independent variables is *not less than the minimum* required to define the system completely, we shall have complete conversion of heat into work, and work into heat. In this case, the equation

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[2] The boldface numbers in parentheses refer to the list of references appended to this paper.

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of state for the internal energy E, written as

$$E = \phi(\alpha_1, \cdots, \alpha_n), \tag{1}$$

where $\alpha_1, \cdots, \alpha_n$ are the n independent variables, is called a *complete* equation of state.

(*b*) If the number of independent variables is less than this minimum number, the equation of state for E will be *incomplete*, and heat cannot be converted completely into work, although work can be converted completely into heat. That is, the Kelvin-Planck and Clausius principles hold only for incompletely described systems, which is the case in practice.

(*c*) Suppose that of the n variables in Eq. 1 the variables $\alpha_1, \cdots, \alpha_i$ are actually observed and controlled, whereas we are not aware of the existence of, or for practical reasons we decide not to use, the remaining variables $\alpha_j, \cdots, \alpha_n$. Consider a change in E given by

$$dE = \left[\frac{\partial E}{\partial \alpha_1} d\alpha_1 + \cdots + \frac{\partial E}{\partial \alpha_i} d\alpha_i\right] + \left[\frac{\partial E}{\partial \alpha_j} d\alpha_j + \cdots + \frac{\partial E}{\partial \alpha_n} d\alpha_n\right]$$
$$= dE_{\mathrm{I}} + dE_{\mathrm{II}}. \tag{2}$$

If in a cyclic process there results a net conversion of heat into work, the writer has shown that

$$\int_{q \to W} dE > 0, \tag{3}$$

where $q \to W$ indicates a process where the net effect is a conversion of heat into work.

In the present paper we wish to analyze the heat, work, and temperature concepts in detail from the same point of view, and shall also examine the temperature scale. The Kelvin absolute temperature scale is based on Carnot cycle, and the perfect gas is the thermometric system which is used for establishing this temperature scale. Actual gaseous systems deviate somewhat from the perfect gas law, and so correction terms have to be applied to the observations. This temperature scale, it is claimed, has the desirable characteristic that it is independent of the nature of the material system which is used for the Carnot cycle. However, the writer has shown (1) that there are serious difficulties in the use of Carnot cycle when a single phase system is subject to more than two independent variables. For example, the efficiency of a Carnot cycle operating between two fixed temperatures is not necessarily unique, but may have many arbitrary values. Then, again, from a practical standpoint, it is desirable to use for our thermometric system a gas which deviates as little as possible from the perfect gas law. If not, the correction terms for reducing our observations to the Kelvin scale become too large and the accuracy very poor. The fact that Kelvin's scale is theoretically independent of the nature of the material

system is no great comfort to the experimenter, for he cannot use systems where the temperature is measured by electric resistance, or by thermo-electric emf., or magnetic susceptibility, and reduce the results directly and rigorously to the Kelvin absolute scale. Practically, he is still restricted to the so-called permanent gases, and above all to systems which are subject to only two independent variables. At the very low temperatures, where matter manifests many unusual characteristics, it would seem to the writer that a rational temperature scale would be of extreme importance, and one which is lacking at present.

2. WORK AND HEAT CONCEPTS FOR COMPLETE AND INCOMPLETE SYSTEMS.

Let us assume that the internal energy E is defined *completely* by Eq. 1. On the basis of the first law of thermodynamics we must assume E to be a single-valued point function of the generalized space $(\alpha_1, \cdots, \alpha_n)$. The total differential will be

$$dE = \frac{\partial E}{\partial \alpha_1} d\alpha_1 + \cdots + \frac{\partial E}{\partial \alpha_n} d\alpha_n = -(A_1 d\alpha_1 + \cdots + A_n d\alpha_n), \quad (4)$$

where $A_p = -(\partial E/\partial \alpha_p)$ by definition. E is then a potential function and the *complete* system is conservative (see reference (1), Section 6). Each A will, in general, be a function of $\alpha_1, \cdots, \alpha_n$. Some of the α's will be configuration coordinates, so that the corresponding A's will be mechanical forces. Other α's may be electric charge and magnetic flux, so that the corresponding A's may be the electric and magnetic field intensities. In addition to these, some α's may be the degrees of transformations, with their corresponding A's as the latent heats; also, some α's may be the masses of the chemical components, and the corresponding A's the chemical potentials of Gibbs.

To the experimenter ϕ is complete only when he is able in some way to observe and measure each α which enters into Eq. 1, although we may consider in theory the consequences of certain α's being present. The terms of Eq. 4 which have reference to mechanical and electromagnetic quantities we shall, as is customary, refer to as work and the corresponding A's as generalized forces. Those terms of Eq. 4 which refer to inter- and intra-phase transformations we shall refer to as (latent) heat. Those involving masses and chemical potentials may be classified as one or the other, depending on which is more prominent. It is interesting to note the following: since each A is, in general, a function of all α's, it is possible to realize changes in mechanical and electrical properties by chemical changes, and *vice versa*. Also, it is possible to realize changes in electric and magnetic properties by mechanical changes, and *vice versa*. All these various types of phenomena are actually observed, and if the system is complete in the sense defined by Eq. 1 the conversion of one form of energy into another will also be complete.

Suppose, however, that the system is *incomplete*—meaning that only

the variables $\alpha_1, \cdots, \alpha_i$ are observed and measured, whereas the variables $\alpha_j, \cdots, \alpha_n$ have been neglected either unconsciously because we are not aware of their existence, or because we know of no means whereby to measure them; or for practical reasons we decide to neglect them. Whatever the reasons may be, we have no information on the magnitudes and changes in $\alpha_j, \cdots, \alpha_n$ and $A_j, \cdots, A_n$. Should we attempt to form a closed cycle by varying only the quantities $\alpha_1, \cdots, \alpha_i$ and finally returning them to their original values, in general the quantities $\alpha_j, \cdots, \alpha_n$ and their corresponding A's will not return to their original values. Furthermore, since $A_1, \cdots, A_i$ are, in general, each a function of all n independent variables, when we return only the quantities $\alpha_1, \cdots, \alpha_i$ to their original values we do not thereby necessarily return $A_1, \cdots, A_i$ to their original values *entirely*. We see then that for an incomplete system the energy represented by

$$-\sum_{p=j}^{n}\int A_p d\alpha_p \tag{5}$$

can no longer be recovered as work, or latent heat of a specific transformation. Also, of the total energy represented by

$$-\sum_{p=1}^{i}\int A_p d\alpha_p \tag{6}$$

some portion will, in general, not be recoverable. In going through the apparent closed cycle the integrals of Eq. 6 will not necessarily vanish completely. Let us write

$$-\delta\sum_{p=1}^{i}\int A_p d\alpha_p \tag{7}$$

for those portions of the energy (work and latent heats) which remain with the system as a result of the apparent closed cycle. The total energy remaining with the system as the result of the apparent closed cycle may then be written as

$$q' = -\delta\sum_{p=1}^{i}\int A_p d\alpha_p - \sum_{p=j}^{n}\int A_p d\alpha_p \tag{8}$$

and this is the energy which produces the "heating up" of the system and is responsible for the inefficiency of the process.

For the remaining portions of the energy of this cycle we have

$$q - W = -\sum_{p=1}^{i}\int A_p d\alpha_p + \delta\sum_{p=1}^{i}\int A_p d\alpha_p, \tag{9}$$

where q is the heat added to the system and W the work done by the system. For this apparent closed cycle we have, then

$$q - W = q' \qquad \text{or} \qquad q = W + q'. \tag{10}$$

Let us consider here a few examples. First, for a single gaseous phase we should write, according to classic thermodynamics,

$$E = \phi_1(p, v) = \phi_2(p, T). \tag{11}$$

In carrying the system through a closed cycle at a finite rate, we find experimentally that the system "heats up," and this effect is greater for, say, ammonia vapor as compared to that for helium or hydrogen. We have actually neglected a variable, and to make Eq. 11 complete (or more complete) we should add the degree of transformation x. As the writer has shown (2), this variable x, by its absence, accounts for metastable states and such states are more pronounced in the case of ammonia as compared to that for helium or hydrogen.

A second example may be a plastic, such as rubber. In this case we have many coefficients of elasticity which are independent of each other, so that an equation of the form $E = \phi(s, l)$, where s is the linear tension and l the linear strain, is surely incomplete. In addition to the coefficients of elasticity we might, under certain circumstances have to add x's to describe chemical or physical transformations.

A third example may be the hysteresis cycle of, say, a piece of steel. By writing $E = \phi(H, B)$, where H is the magnetic field intensity and B the magnetic flux, we have neglected mechanical, crystalline and possibly other essential properties of the system, and we know from experience that the energy of a hysteresis cycle depends on the past history as well as on H, B.

3. ACTUAL AND QUASISTATIC PROCESSES.

Let us now examine the significance of an actual process which takes place at a finite rate, and that of a quasistatic process.

Equilibrium was considered in a previous paper (1). Assuming that n independent variables define our system completely, we may say that when all α's are held constant with respect to time, no work of any sort can be done by or on the system, for the α's include all the configuration coordinates. Furthermore, there will be no internal changes such as chemical transformations (inter- and intra-phase changes) or changes in mechanical, electric or magnetic properties. We may then define *true equilibrium* as that state of the system where $\alpha_1, \cdots, \alpha_n$ are constant with respect to time.

For a complete system there is necessarily no difference whether changes in the α's are made very slowly or at a finite rate, for in either case the resulting change in E is clearly defined and of the same value. But, now let us consider an incomplete system where only $\alpha_1, \cdots, \alpha_i$

are observed and controlled. The quasistatic process calls for changes in $\alpha_1, \cdots, \alpha_i$ that are infinitesimally slow. We are, however, not certain that the variables $\alpha_j, \cdots, \alpha_n$ have not been altered, for they are not observed. If changes in $\alpha_1, \cdots, \alpha_i$ are made at a finite rate we know from experience that the cycles are not closed and that they depend more or less on the past history of the system. From the above considerations this is understandable because, since $A_1, \cdots, A_i$ are functions of all α's, then if $\alpha_j, \cdots, \alpha_n$ vary in some unknown way, at the beginning of each successive cycle we start with different values of $A_1, \cdots, A_i$. It is conceivable that changes in $\alpha_j, \cdots, \alpha_n$ may be very small, or even nil, when the changes in $\alpha_1, \cdots, \alpha_i$ are infinitesimally slow. However, that must be proven for each particular system, and cannot be taken as a generalization. We have no basis for assuming that a process is reversible only when it is quasistatic. Thus the conception of a quasistatic process, even in theory, does not overcome the difficulties encountered by an incomplete system. In fact, this concept adds serious restrictions to thermodynamics for it limits our study only to static systems.

As an alternative it would appear to the writer that the study of systems solely from the point of view of degree of completeness is clearer and much more informative. We must recognize the fact that certain indeterminacies which exist with systems are unavoidable, and can be diminished only as we derive more knowledge of the systems. From our present knowledge of systems it would appear that the rarer the system the more completely could it be defined, as, for example, rarified charges in an electromagnetic field, or rarified matter in a gravitational field. Such systems are conservative. Cycles involving gases at low pressures are more "efficient" than cycles involving liquids, and the latter are more "efficient" than cycles involving solids. Also, since crystalline media are more clearly defined than plastics it would appear that cycles involving crystalline solids are more "efficient" than those involving plastics.

4. THE "TRUE" TEMPERATURE SCALE.

We shall assume that Eq. 1 defines the internal energy of a thermodynamic system completely. As regards temperature, we shall assume that we know in a general way (empirically) what this means. Temperature may then be one of the independent variables of the system. To be able to do this, however, we should have to establish a temperature scale which is independent of the other α's of the system under consideration; in other words, temperature must be defined on some scale not related to the system.

It should be noted that the essential requirement for an equation of state to be complete is that the number n of independent variables shall be *not less than the minimum required to define the system completely.* The

particular choice of variables to be used may be arbitrary and differ with circumstances. We may, for example, desire to write for our temperature

$$T = \psi(\alpha_1, \cdots, \alpha_n), \tag{12}$$

where T will be a *derived* quantity and will depend on the system. The system may then be used as a thermometer to define its particular temperature scale. On this scale, temperature will be a function of, say, α_1, *only if* $\alpha_2, \cdots, \alpha_n$ *are held constant.* In practice, when we state that T is a function of (p, v) for a fluid, or of the electric resistance, or of the thermoelectric emf., or of the magnetic susceptibility, we employ incomplete relations, for on close examination we may find other independent quantities that should have been introduced into the equation for each particular system. In other words, in practice we do not use all required α's to define T, and for that reason each system will, by itself, establish a temperature scale which, in general, will be different from the others.

Suppose we consider two systems, A and B, each to be defined completely by the same set of n independent variables. Should one system require a lesser number of variables, we may still use n for both in our formal considerations, bearing in mind that some of these variables may be of constant value or zero. If A is defined completely by $\alpha_1, \cdots, \alpha_k$, and B by $\alpha_1, \cdots, \alpha_n$, we may still retain our formal Eqs. 1 and 12 above, recognizing that for B we have $\alpha_1, \cdots, \alpha_k$ constant or zero, and for A we have $\alpha_1, \cdots, \alpha_n$ constant or zero.

If the temperature scale for each system is given by a ψ function, for equality of temperature let us write

$$T = \psi_A(\alpha_1, \cdots, \alpha_n) = \psi_B(\alpha_1, \cdots, \alpha_n). \tag{13}$$

Assuming that A and B are any two systems of the entire class whose equations of state are complete, we may utilize each of these systems to establish a temperature scale, and with Eq. 13 all of these scales will be identical. Let us refer to this as the "true" temperature scale.

Is the Kelvin absolute scale a "true" temperature scale? The Kelvin scale is entirely restricted to a two-variable system, both from the standpoint of Carnot cycle and from the perfect gas law. We know from experience that no material system, including a single gaseous phase, is completely defined by only two independent variables (2). Therefore, Kelvin's scale rests on a hypothetical system where, say, in Eq. 12, $\alpha_1 = p$, $\alpha_2 = v$, and $\alpha_3 = \cdots = \alpha_n = 0$. Since actual systems do not meet these requirements, we should be inclined to state that the Kelvin scale is not a "true" temperature scale in the sense defined by Eqs. 12 and 13.

Logically there appears to be no objection to adopting Kelvin's as our standard temperature scale. In doing so, temperature will be an independent quantity, not derived from a complete equation for an

actual system. Kelvin's scale will be independent of material systems in the sense that further knowledge of the behavior of systems, which may lead to a more complete equation of state, will not yield additional information nor accuracy with regard to this scale. Kelvin's scale fixes the absolute zero, and from the classic standpoint this scale should have no negative values.

The objections to Kelvin's scale would come mostly from practical considerations, but those are quite serious. In spite of the fact that Kelvin's scale is designed for a perfect gas, we must actually work with gases which are more or less imperfect. At the very low temperatures, such as the liquefaction point of helium, our working gas is very imperfect; at still lower temperatures we may not have any gas to work with. We certainly cannot interpret directly and rationally (except by extrapolation) measurement of, say, magnetic susceptibility into temperature on the Kelvin scale. At the very low temperatures we are forced, for practical reasons, to consider T as a derived quantity, meaning that we must resort to an incomplete equation (that is, Curie's law).

At the higher temperatures, say above room temperature, objections may be raised as to the way observations on actual gaseous systems are reduced to Kelvin's temperature scale. These objections may be based on the fact that it is assumed that $T = f(p, v)$ for the gas, whereas actually it should be a function of more than two variables (2).

Should we decide to use the "true" temperature scale in our investigations, we should of course also encounter difficulties, for all equations of state which are used in practice are incomplete. That is the basis for the validity of Kelvin-Planck and Clausius principles (1). In practice we do not use or know all n independent variables, and our systems are studied under conditions in which uncertainties do exist. It would appear then that there must be uncertainties in our evaluation of the "true" temperature T. However, the experimenter is always confronted with inaccuracies in his measurements, and intuitively he seeks more and more controls (or more and more α's) in order to make his measurements more reproducible and more accurate. As the science of thermometry progresses we should expect, then, that "true" temperatures as measured by different systems will come closer and closer in agreement, whereas Kelvin's scale is set once for all time and very little more can be done for it. For these reasons the writer would be inclined to favor our considering temperature as a derived quantity based on Eq. 12 and to define our scale on the basis of Eqs. 12 and 13.

5. CARNOT CYCLE AND THE INCOMPLETE EQUATION OF STATE.

We shall consider a system whose state may theoretically be completely described by Eqs. 1 and 12, but in practice we shall operate it only as an incomplete system. Let us assume that $\alpha_1 = p$, $\alpha_2 = v$ (that is, $i = 2$) are the variables which are observed and controlled.

We shall carry this system through an apparent closed cycle in a manner such that the net heat absorbed by, and the net work done by the system are both positive. As far as possible we shall try to make this a Carnot cycle, and shall proceed as follows.

Assume that two large heat reservoirs are available, each of which is kept at a constant temperature as measured on some arbitrary scale which we shall indicate by θ. Let the first reservoir be at the temperature θ_1, and the second at θ_2, where $\theta_1 > \theta_2$. We shall make a distinction between the scales θ and T, where the former is to serve only as a means for detecting fluctuations and to aid in keeping the reservoirs at a constant temperature, whereas T is the true temperature scale defined by our system.

Referring to Fig. 1, let us assume that along the path AB the system is in direct contact with the first reservoir. During the time of this

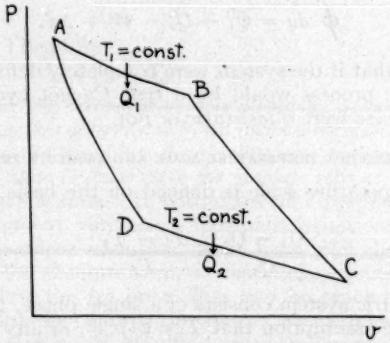

FIG. 1.

contact we change (p, v) of the system, keeping T constant and equal to T_1. Along this path the internal energy will change in value, so that the heat taken by the system from the first reservoir will be, by Eq. 10,

$$\int_A^B dq = Q_1 = \int_A^B dW + \int_{A'}^{B'} dq'. \tag{14}$$

We write the limits for the last integral as A', B' instead of A, B because, in general, there would be some uncertainty in the states defined by the limits of $\alpha_1, \cdots, \alpha_n$, and A' may be somewhat different from A, and B' from B.

From B to C we desire to subject the system to an adiabatic process. For this we write

$$\int_B^C dq = 0 = \int_B^C dW + \int_{B''}^{C''} dq', \tag{15}$$

where the last integral is to take on the uncertainties, and B'' is somewhat different from B', B and C'' somewhat different from C.

Along the path CD we place the system in direct contact with the second reservoir, which if measured by the system is at the temperature T_2. The process along this path is an isothermal, but in the reverse sense to that along AB. The path DA is an adiabatic similar, but in the reverse sense, to BC. Thus,

$$\int_C^D dq = -Q_2 = \int_C^D dW + \int_{C^{III}}^{D^{III}} dq' \tag{16}$$

and

$$\int_D^A dq = 0 = \int_D^A dW + \int_{D^{IV}}^{A^{IV}} dq'. \tag{17}$$

Adding these four equations we have

$$\oint dq = Q_1 - Q_2 = W + \Delta q'. \tag{18}$$

It will be noted that if the system were completely defined by p, v, then $\Delta q' = 0$ and the process would be a true Carnot cycle regardless of whether the process were quasistatic or not.

6. KELVIN'S TEMPERATURE SCALE AND ABSOLUTE ZERO.

Kelvin's temperature scale is defined on the basis of Carnot cycle by writing

$$\frac{Q_1 - Q_2}{Q_2} = \frac{T_1 - T_2}{T_2}. \tag{19}$$

If our thermometric system consists of a single phase, this temperature scale rests on the assumption that $T = \psi(p, v)$, or any other two independent variables. If the system is *complete* on the basis of these two variables, then

$$Q_1 - Q_2 = W \tag{20}$$

and

$$\frac{W}{Q_2} = \frac{T_1 - T_2}{T_2}. \tag{21}$$

However, if the system is incomplete on the basis of two independent variables, we must use Eq. 18 and write

$$\frac{W + \Delta q'}{Q_2} = \frac{T_1 - T_2}{T_2}. \tag{22}$$

Suppose we let T_1 represent the boiling point of water, and T_2 the ice point, both taken under standard conditions. Write

$$T_1 - T_2 = 100. \tag{23}$$

Substituting Eq. 23 into Eq. 21, we have for a complete system

$$T_{2C} = 100\,\frac{Q_2}{W}. \tag{24}$$

Substituting Eq. 23 into Eq. 22, we have for an incomplete system

$$T_{2I} = 100\,\frac{Q_2}{W + \Delta q'}. \tag{25}$$

T_{2C} is the ice point on the temperature scale defined by the complete equation $T = \psi(p, v)$, whereas T_{2I} is the ice point on the temperature scale defined by the incomplete equation $T = \psi(p, v)$, where the complete equation should be $T = \psi(p, v, \alpha_3, \cdots, \alpha_n)$. From Eqs. 24 and 25, we have

$$\frac{T_{2I}}{T_{2C}} = \frac{W}{W + \Delta q'} < 1 \tag{26}$$

since $\Delta q' > 0$. Therefore

$$T_{2I} < T_{2C}, \tag{27}$$

which states that the *ice point as determined by a complete equation of state is higher than that determined by an incomplete equation of state.* Or, with reference to the ice point, *the absolute zero as determined by a complete equation of state is lower than the absolute zero as determined by an incomplete equation of state.* In both cases the equation *as used* involves only two independent variables. Complete equations of state of any number of independent variables, as we have seen above, should yield the same value for absolute zero, so that $-T_{2C}$ is the ultimate lowest temperature below the ice point that can be reached. This is most likely unattainable for the reason that as we examine a given system more and more closely, we may find more and more independent variables and will therefore never attain a complete equation of state.

7. A SUGGESTION FOR A TEMPERATURE SCALE.

We have seen that Kelvin's temperature scale rests on the notion that we have a complete equation of state for a system, and that this equation is made up of just two independent variables for each phase. It is hardly likely that we shall find these conditions satisfied over the entire temperature range, high and very low, even to a moderate degree of accuracy. In view of this, and in the light of the above considerations, the writer wishes to propose a new temperature scale for the reader's consideration. This is, of course, merely a suggestion, subject to further study both from the theoretical and experimental standpoints.

We have seen above that a change in the internal energy dE of a system will, in general, arise from the following three sources: (1) work of a mechanical, electric or magnetic character done by or on the system, which is observable and can be measured; (2) latent heats of trans-

formations and heats of chemical reactions, both of which are observable and can be measured; and, finally, (3) energy which remains with the system due to the incompleteness of the system and which cannot be attributed to any particular form of work or latent heat.

Suppose we subject a system to energy changes in a manner such that no work of any sort is done by or on the system. As a specific example, let us take a cylinder of quartz in which is installed a fine electric heater. We can impart energy (heat) to this system without this energy resulting in any external work. Whatever transformations or chemical changes take place in this system, will be measured by the heat added to the system. In fact, the system may be incomplete as far as our knowledge of it is concerned, and still the heat added to the system will measure *all* energy, provided that the work is zero. In this way we can treat the system as complete and accurately measure its total energy changes. All necessary experimental precautions, such as enclosing the system in a vacuum and surrounding it with radiation shields, may be taken to prevent heat losses, and we can then write for the complete system $dE = dQ$, with $dW = 0$, where dQ is the measureable heat input.

Let us, then, define the new temperature scale θ by writing the following:

$$dE = dQ = Md\theta, \tag{28}$$

where dE is the increase in the internal energy when $dW = 0$, dQ is the heat added, and M the total mass of the system under consideration. From classic thermodynamics we should have the relation

$$d\theta = C_v dT, \tag{29}$$

where C_v is the specific heat at constant volume, and T the temperature on the Kelvin scale. Since C_v is not constant, θ will not necessarily be proportional to T; however, C_v is always positive, so that an increase in T is always accompanied by an increase in θ. In expressing $d\theta$ as proportional to dQ we do nothing which is essentially new, for we see that in Eq. 19 Kelvin makes a similar assumption. The difference, of course, rests in the treatment of dW.

To calibrate such a thermometric system let us proceed as follows: First allow the system to come to equilibrium in contact with melting ice under standard conditions, and assign to it the arbitrary state E_0, θ_0. Then place this system into a carefully shielded vacuum space and apply heat to the system until a differential thermocouple indicates no temperature difference between it and steam under standard conditions. The heat, Q_1, added to the system will satisfy the relation

$$Q_1 = E_{100} - E_0 = M(\theta_{100} - \theta_0), \tag{30}$$

where E_{100}, θ_{100} represent the state of the system at the steam point.

In a second experiment we may start at the ice point and apply heat

Q until the system reaches a temperature θ defined by

$$Q = E - E_0 = M(\theta - \theta_0). \tag{31}$$

Dividing Eq. 31 by Eq. 30, we have

$$\frac{Q}{Q_1} = \frac{E - E_0}{E_{100} - E_0} = \frac{\theta - \theta_0}{\theta_{100} - \theta_0}. \tag{32}$$

Suppose we place

$$\theta_{100} - \theta_0 = 100. \tag{33}$$

Then,

$$\theta = \theta_0 + 100\frac{Q}{Q_1}, \tag{34}$$

which is the fundamental equation for θ. Since all n variables of the system are taken into account, we operate as though the system were

TABLE I.
Values of θ.

°C.	Corundum	Quartz	Cristobalite	Vitreous Quartz	Gold	Platinum	Graphite
−200	...	−116.6	−118.2	−119.0	...	...	...
−100	...	−75.5	−75.7	−76.3	...	...	...
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
100	100.0	100.0	100.0	100.0	100.0	100.0	100.0
200	217.9	217.5	223.4	216.3	201.9	202.1	233.2
300	347.3	348.5	364.5	343.9	305.7	306.3	393.0
400	483.5	489.3	498.6	480.2	411.5	412.6	574.2
500	624.5	638.6	639.0	624.9	...	521.1	772.4
600	769.2	812.6	782.7	772.3	...	631.6	984.0
700	917.0	958.8	931.6	923.7	...	744.2	1208.4
800	1087.2	1106.0	1081.8	1077.4	...	858.9	1435.8
900	1219.3	1253.5	1235.1	1231.9	...	...	...
1000	...	1404.4	1388.7	1390.7	...	...	...
1100	...	1557.5	1544.2	1546.9	...	...	...
1200	...	1709.9	1698.2	1707.7	...	...	...
1300	...	1864.1	1852.8	1876.9	...	...	...
1400	...	2021.8	2009.4	2044.8	...	...	...

complete. For this reason it is to be expected that the θ-scale will be independent of the system that is selected, just so long as the generalized work done by or on the system is zero throughout the range of operation.

With respect to an absolute zero the value of the ice point, θ_0, can be obtained from Eq. 34 by placing $\theta = 0$; thus

$$\theta_0 = 100\frac{Q_0}{Q_1}, \tag{35}$$

where Q_0 is the heat added to the system in going from $\theta = 0$ to $\theta = \theta_0$. Substituting Eq. 35 into Eq. 34, we have

$$\theta = 100\frac{Q + Q_0}{Q_1}, \tag{36}$$

which yields our temperature scale with respect to absolute zero.

To investigate the possibilities of such a temperature scale, the writer has selected some published data on systems which might be considered suitable for our purpose, that is, where $dW = 0$. The data on α-Al_2O_3 (corundum) were obtained by D. C. Ginnings and R. J. Corruccini (4). The data on quartz, cristobalite, vitreous quartz, gold, platinum and graphite are those given in the International Critical Tables (5). Where the average specific heats were given the total heat $Q = \int_0^T C_m dT$ was calculated. For each substance a value of Q_1 was obtained, and then values of θ were calculated on the basis of the equation $\theta = 100 \frac{Q}{Q_1}$, taking $\theta_0 = 0$. In Table I are given the values of θ thus calculated for different Centigrade temperatures.

Even though there are considerable discrepancies in the values of θ as determined from the different substances at the same Centigrade temperature, as a first trial there appear to be possibilities. Undoubtedly the greatest source of error is in the value of Q_1. It would certainly be of great interest to investigate this more fully in the laboratory.

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DIMENSIONAL ANALYSIS: AN APPROACH FROM TRANSFORMATION THEORY AND A CRITERION FOR SCALING MODEL EXPERIMENTS.[1]

BY

J. C. DECIUS.[2]

1. THE INVARIANTS OF CONTINUOUS TRANSFORMATION GROUPS.

In a recent article in this journal, Langhaar (1)[3] has drawn attention to a method of dimensional analysis in which the kinds of physical quantities are explicitly given a vector representation. The crucial theorem of this subject was proved by Langhaar in terms of theorems on homogeneous functions.

The author of the present paper has formalized the subject matter of dimensional analysis from a slightly different mathematical point of view, in which, however, as is natural, the material receives a practically identical representation. Because of the relationship of the following method to a very general method of discussing problems in mathematical physics, it seems worth-while to set it forth here. In addition, since the development is aimed at the treatment of scaling laws for model experiments, a new criterion is given for the possibility of satisfying the scaling laws when certain restrictions are imposed upon the variables.

A fruitful approach to the problems of dimensional analysis may be made by regarding any physical equation as the expression of an invariant under the transformations of some group. Frequently the knowledge of some form of spatial symmetry, the requirement of invariance under permutation of indistinguishable particles, or the necessity of special invariant properties which must obtain under transformations simultaneously involving temporal and spatial coordinates has been used to simplify or actually to advance the mathematical description of the physical world. Although the results of the following application of invariant theory are all rather well known, the method itself is rather elucidating, particularly with regard to the derivation of the laws of similitude which govern the scaling of model experiments.

It will be shown below that all of the results of dimensional analysis follow from the single postulate that all physical relations must be expressible in a form which does not depend upon the "magnitude" of the various physical units of measurement requisite for the description of a given physical situation.

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[1] Contribution No. 393 from the Woods Hole Oceanographic Institution.

[2] Woods Hole Oceanographic Institution, Woods Hole, Mass.; now at Brown University, Providence, Rhode Island.

[3] The boldface numbers in parentheses refer to the list of references appended to this paper.

Such changes of unit magnitude form the representation of a group of continuous transformations. A group of continuous transformations may be represented by the equations:

$$\begin{aligned} \bar{x}^i &= \xi^i(x^1, x^2, \cdots x^n; \quad \lambda^1, \lambda^2, \cdots \lambda^m), \\ i &= 1, 2, \cdots n \end{aligned} \tag{1.1}$$

in which the barred quantities are the new values of the variables, which were originally $x^1, x^2, \cdots x^n$, after the transformation induced by the set of continuously variable, independent parameters, $\lambda^j (j = 1, 2, \cdots m)$. The usual postulates for a group require that the λ^j can assume such values as will:

1. Induce the identity transformation

$$x^i = \xi^i(x^1 \cdots x^n; \quad \lambda_0{}^1 \cdots \lambda_0{}^m).$$

2. Induce the resultant of two or more successive transformations in one step, that is, if

$$\begin{aligned} \bar{x}^i &= \xi^i(x^1 \cdots x^n; \quad \lambda_1{}^1 \cdots \lambda_1{}^m), \\ \bar{\bar{x}}^i &= \xi^i(\bar{x}^1 \cdots \bar{x}^n; \quad \lambda_2{}^1 \cdots \lambda_2{}^m), \end{aligned}$$

then

$$\bar{\bar{x}}^i = \xi^i(x^1 \cdots x^n; \quad \lambda_{12}{}^1 \cdots \lambda_{12}{}^m).$$

3. Induce a transformation "inverse" to every given transformation, that is, if

$$\begin{aligned} \bar{x}^i &= \xi^i(x^1 \cdots x^n; \quad \lambda_1{}^1 \cdots \lambda_1{}^m), \\ x^i &= \xi^i(\bar{x}^1 \cdots \bar{x}^n; \quad \lambda_{-1}{}^1 \cdots \lambda_{-1}{}^m). \end{aligned}$$

It may then be shown that the equations of transformation (1.1) determine a set of linear partial differential operators

$$U_j = \left(\frac{\partial \xi^i}{\partial \lambda^j}\right)_0 \frac{\partial}{\partial x^i} \tag{1.2}$[4]$$

which represent the independent infinitesimal transformations of the group (the subscript zero for $\partial \xi^i / \partial \lambda^j$ implies that the derivative is to be evaluated for those values of the parameters, λ^j, which induce the identity transformation: without loss of generality these values may henceforth be assumed to be zero). The most general infinitesimal transformation is represented by

$$U = \alpha^j U_j \tag{1.3}$$

in which the α^j are arbitrary constants; any finite transformation is

[4] The summation convention of tensor notation is used throughout. Thus $\left(\frac{\partial \xi^i}{\partial \lambda^j}\right)_0 \frac{\partial}{\partial x^i}$ implies $\sum_{i=1}^{n} \left(\frac{\partial \xi^i}{\partial \lambda^j}\right)_0 \frac{\partial}{\partial x^i}$; this is true only if the index appears in one place as a superscript, in another place as a subscript.

generated by an integration of the differential operation signified by (1.3).

The important consequence for the present purposes is that if any function of the variables x^i, say

$$F(x^1, \cdots x^n) = 0, \tag{1.4}$$

is known to be invariant under the group represented by (1.1), then it is always possible to express (1.4) in terms of a complete set of fundamental invariants $y^1, y^2, \cdots y^{n-r}$, which are determined by

$$Uy = 0 \tag{1.5}$$

as functions of the x^i. Thus

$$F(x^1, \cdots x^n) = \Phi(y^1, \cdots y^{n-r}) = 0 \tag{1.6}$$

and the number of independent variables has been reduced by r, which is equal to the rank of the matrix $\left\| \left(\frac{\partial \xi^i}{\partial \lambda^j} \right)_0 \right\|$.

2. DETERMINATION OF THE INVARIANTS OF A GENERAL PHYSICAL EQUATION UNDER TRANSFORMATIONS OF THE UNITS.

The nature of the number, x^j, resulting from a physical measurement of the jth kind of physical quantity is such that it is determined only within a transformation of the type:

$$\bar{x}^j = \exp [\lambda^j] x^j, \qquad -\infty < \lambda^j < +\infty. \tag{2.1}$$[5]

If it is desired to deduce the possible kinds of functions of the *magnitudes* of a basic set of physical quantities, which give the *magnitude* of a resultant physical quantity derivable from the basic ones, it is possible to apply the principles of Section 1 in the following manner. The resultant magnitude itself must obey a transformation law of the same form as (2.1)

$$\bar{x} = \exp [f(\lambda^j)] x \tag{2.2}$$

in which the value of f may be taken as zero when $\lambda^j = 0$, all j. Then Eq. 1.3 requires that the relation between x and x^j be expressible in terms of a function y determined by

$$\alpha^k \left[\delta_k{}^j x^k \frac{\partial y}{\partial x^j} + \left(\frac{\partial f}{\partial \lambda^k} \right)_0 x \frac{\partial y}{\partial x} \right] = 0. \tag{2.3}$$

The α^k being arbitrary constants, (2.3) is equivalent to the m relations

$$\frac{\partial y}{\partial \ln x^k} = -\left(\frac{\partial f}{\partial \lambda^k} \right)_0 \frac{\partial y}{\partial \ln x}, \qquad k = 1, 2, \cdots m. \tag{2.4}$$

[5] It is clear that the range of the λ^j indicated will contain values satisfying all the group postulates of Section 1.

An acceptable solution of this system is

$$y = \prod_{k=1}^{m} \frac{(x^k)^{a_k}}{x}, \tag{2.5}$$

where $a_k = \left(\frac{\partial f}{\partial \lambda^k}\right)_0$ is a constant, so that, y being actually an invariant, which can be assumed to have the numerical value of unity, the acceptable form for the combination of magnitudes is

$$x = \prod_{k=1}^{m} (x^k)^{a_k} \tag{2.6}$$

with the transformation law:

$$\bar{x} = \exp\,[a_k \lambda^k] x. \tag{2.7}$$

It is immediately perceived that the numbers a_k characterize the *kind* of the derived quantity and that they may be regarded as the components of an m-dimensional vector in the basis in which the lth component of the kth fundamental quantity is the Kronecker symbol, $\delta_l{}^k$. In other words, the general symbol, $a_j{}^i$, is the ordinary exponent occurring in the conventional dimensional formula for the ith *kind* of variable relative to the jth fundamental *kind* of quantity; for example, let $i = 1$ designate energy; $j = 1, 2, 3$ correspond to mass, length, and time, respectively, then $a_1{}^1 = 1$, $a_2{}^1 = 2$, $a_3{}^1 = -2$.

Now let x^i be a set of derived magnitudes expressible in terms of the basic set, x^j. Then the transformation laws are:

$$\begin{gathered} \bar{x}^i = \exp\,[a_j{}^i \lambda^j] x^i, \\ i = 1, 2, \cdots n; \quad j = 1, 2, \cdots m. \end{gathered} \tag{2.8}$$

Now let

$$F(x^i) = 0$$

be the expression of an unknown physical relation in terms of the magnitudes, x^i. Equations 1.2, 1.3, and 1.5 require that the relation be expressible in terms of y^k, the solutions of

$$a^i a_j{}^i x^i \frac{\partial y^k}{\partial x^i} = 0$$

or

$$a_j{}^i \frac{\partial y^k}{\partial \ln x^i} = 0 \tag{2.9}$$

$$\text{for} \quad j = 1, 2, \cdots m.$$

Solution of Eq. 2.9 gives

$$y^k = \prod_{i=1}^{n} (x^i)^{b_i^k}, \tag{2.10}$$

$$k = 1, 2, \cdots n - r,$$

where the b_i^k are constants determined by

$$a_j^i b_i^k = 0 \tag{2.11}$$

and where r is the rank of the matrix $\|a_j^i\|$.

This result is, of course, the well-known "π-theorem" as described by Buckingham (2) or Bridgman (3). The virtue of having used the transformation theory lies chiefly in the following two points:

1. The "alias-alibi" duality of any transformation immediately allows the conclusion that although the deductions were based on the assumption that the "absolute" magnitude of the physical quantities were unchanged during the transformation, the invariants, y^k, will certainly satisfy the equation

$$\Phi(y^k) = 0$$

throughout a transformation in which the fundamental magnitudes are fixed and the magnitudes of the x^i are "actually" changed but in such a way as to keep the y^k constant. In other words, empirical physical information satisfying

$$F(x^i) = 0$$

on a given *scale* may be used to predict a physical relation on any other scale, provided only that the x^i are varied in such a way as to maintain the constancy of the y^k.

2. The simplification of a physical problem brought about by the reduction of the degrees of freedom, n, by the number r, is not mathematically different in kind from the complete solution of the problem (the determination of the form of the unknown function, F) which is obtained when $(n - 1)$ independent transformation parameters are obtained.

3. A FORMAL SOLUTION FOR b_i^k.

Equations 2.11 are equivalent to the matrix equation

$$AB = 0, \tag{3.1}$$

where $A = \|a_j^i\|$, $B = \|b_i^k\|$. The rank of A is r; by mere rearrangement of rows and columns, the non-vanishing determinant can be made to appear in the upper right-hand corner of A (under the last r columns and the first r rows; it will be supposed that the subscript is the row index, the superscript the column index). Call this portion of the matrix A_1, call the first $(n - r)$ columns and first r rows A_2; let the last $(m - r)$ rows be called A_0 (if $r = m$, A_0 does not exist).

Make a corresponding partition of B, calling the first $(n - r)$ rows

B_2, the remaining rows B_1. Then Eq. 3.1 is equivalent to

$$\left\| \begin{array}{c|c} A_2B_2 + A_1B_1 & \\ \hline A_0 & \begin{array}{c} B_2 \\ \hline B_1 \end{array} \end{array} \right\| = 0. \tag{3.2}$$

But since the rows of A_0 are linearly dependent upon the rows of $\|A_2 \vdots A_1\|$, any solution of

$$A_1B_1 = -A_2B_2 \tag{3.3}$$

automatically makes $A_0 \left\| \begin{array}{c} B_2 \\ \hline B_1 \end{array} \right\|$ vanish. Since $|A_1| \neq 0$, A_1^{-1} exists and

$$B_1 = -A_1^{-1}A_2B_2. \tag{3.4}$$

The elements of B_2 are now completely arbitrary, and it is convenient, as will be shown below, to let B_2 equal the negative unit matrix, whence

$$B_1 = A_1^{-1}A_2. \tag{3.5}$$

Combination with Eqs. 2.10 and 2.11 gives the result that the invariants are:

$$y^k = \prod_{p=1}^{r} \frac{(x^{n-r+p})^{b^k_{n-r+p}}}{x^k}, \tag{3.6}$$

that is, since the y^k are to be held constant, the first $(n - r)$ variables must be individually proportional to a set of products involving only the last r variables which may be changed arbitrarily.

4. THE SCALING CRITERION.

In the application of these results to the design of model experiments certain difficulties arise which have not previously been considered in the general case. It frequently occurs that some of the variables essential to a given problem may not be readily changed with scale. As examples, the acceleration of gravity, or, in certain cases, even the properties of liquid and solid media cannot be readily altered in general so as to satisfy Eqs. 3.6 and must therefore be regarded as fixed. In order to set up a general criterion for determining whether scaling is possible, it will be useful to classify the variables in three types: those which are fixed, those which are to be arbitrarily (and independently) scaled, and those which are unrestricted. We shall use the subscripts f and s to designate the first two types respectively; r_s, r_f, r_{sf} will stand for the rank of the sub-matrix of A corresponding to all the variables of types s, f, and both s and f, respectively: t_s, t_f, t_{sf} will stand for the number of y^k which involve variables of type s, f, s and/or f, respectively; and n_s, n_f are the numbers of variables of type s and of type f. A neces-

sary and sufficient condition that scaling be possible is that

$$n_s + r_f = r_{sf}. \tag{4.1}$$

To prove the necessity of the condition, suppose $n_s + r_f \neq r_{sf}$. Since r_{sf} cannot exceed $n_s + r_f$, we must suppose $n_s + r_f > r_{sf}$ or $n_s + n_f - r_{sf} > n_f - r_f$, but since

$$t_{sf} = n_s + n_f - r_{sf}$$

and

$$t_f = n_f - r_f$$

we should have

$$t_{sf} > t_f$$

so that there would be either or both of the following types of y^k:

1. those containing arbitrarily scaled variables only,
2. those containing arbitrarily scaled and fixed, but not unrestricted variables,

neither of which types can be allowed.

To prove that the condition is sufficient, distinguish the two possible cases:

1. $r_{sf} = r$
2. $r_{sf} < r$.

In case (1), that part of A_{sf} with the $r_{sf} \times r_{sf}$ non-vanishing determinant can be taken as A_1; consequently (since such a determinant can be found which contains columns from all the arbitrarily scaled variables) all the arbitrarily scaled variables will appear in the set x^{n-r+p}, the fixed variables (if any) not contained in A_1 will be expressible in terms of fixed variables in A_1 only, and there will be just enough restricted variables to complete the set of $(n - r)y^k$'s with one such variable to each y^k involving arbitrarily scaled variables.

If, on the other hand, case (2) obtains, the same proof will hold (with the modification that some unrestricted variables will appear in the set x^{n-r+p}) provided only that the $r_{sf} \times r_{sf}$ non-vanishing determinant contained in A_{sf} is contained in some $r \times r$ non-vanishing determinant of A.

That this latter requirement is always satisfied follows, for example, from the definition of rank in terms of linear independence as shown by Birkhoff and MacLane (4). This completes the proof of the scaling criterion.

In concluding this section it should be remarked that the criterion furnished by Eq. 4.1 is independent of the basis of fundamental units adopted, since the only quantities appearing are numbers of variables and the rank of various sub-matrices, taken by columns, which are, of course, all invariant under the group of homogeneous linear transformations on the a_j^i, $A \to \bar{A} = TA$, $|T| \neq 0$, which corresponds to all conceivable choices of a basic set of fundamental units.

5. APPLICATION TO HYDRODYNAMICS.

Consider the dimensional composition of the variables appearing in the analytical description of fluid flow. The Navier-Stokes equation and the equation of continuity contain variables of the following dimensional types: v = velocity, p = pressure, R = any length, t = time, ρ = density, μ = viscosity. If the external forces are due to gravity, the acceleration of gravity, g, is included. In addition, there exists an equation of state which expresses ρ as a function of p alone in case the flow is assumed to be isothermal or adiabatic. The implication of this last condition is important in the consideration of possible scaling solutions: although the specific form of the equation of state is unknown, it must be expressible in terms of variables, such as the bulk modulus, whose columns in A are linear combinations of the columns in A representing ρ and p. In order to maintain, temporarily, the generality of the discussion, it will be supposed that the density is given by the series:

$$\rho = \sum_{i=\alpha}^{\omega} \rho_i (p)^i \tag{5.1}$$

with $\alpha \leqq \omega$.
The A matrix is then as follows:

A	v	p	t	ρ_i	μ	g	R
M	0	1	0	$1-i$	1	0	0
L	1	-1	0	$-3+i$	-1	1	1
T	-1	-2	1	$2i$	-1	-2	0

A model experiment will now be considered in which R plays the role of the deliberately scaled variables: $n_s = 1$. Then the criterion of Eq. 4.1 requires $r_{sf} = r_f + 1$. If the medium is unchanged with scale, in the general case the rank of (ρ_i, μ) is 3 so that, since r_{sf} also equals 3, scaling is impossible. Even for an incompressible fluid ($\rho_i = 0$ except for $i = 0$) a scaling solution with fixed g is impossible if μ is fixed. The idealizations which lead to the familiar scaling approximations in terms of the Froude, Reynolds, and Mach numbers are described in Table I. In each case the set of variables designated as "ignored" are shown to be

TABLE I.

Scaling Laws for Hydrodynamics.

Typical Invariant	Ignorable Variables	v	p	t	ρ_i	μ	g	Fixed Variables
Froude Number	$\rho_i (i \neq 0), \mu$	$\frac{1}{2}$	1	$\frac{1}{2}$	$(-i)$	$(\frac{1}{2})$	0	ρ_0, g
Reynolds Number	$\rho_i (i \neq 0), g$	-1	-2	2	$(2i)$	0	(-3)	ρ_0, μ
Mach Number	μ, g	0	0	1	0	(1)	(-1)	ρ_i
—	$\rho_i (i \neq 0, 1), \mu$	$\frac{1}{2}$	0	$\frac{1}{2}$	$[-1]$	$(\frac{1}{2})$	0	$\frac{\rho_i}{\rho_0}, g$

required to follow rather impractical scaling laws, but by assuming that their influence on the flow is unimportant in a certain range of velocities, a solution is obtained by omitting them from further consideration. The dependence of the variables is expressed in terms of the exponent b_i^R in the expression

$$x^i = x^{io}\left(\frac{R}{R^o}\right)^{b_i^R}, \tag{5.2}$$

where x^i is the appropriate value of the ith variable on any scale with a typical length R related to the model scale with corresponding values x^{io} and R^o. The computation of the b_i^R requires only the determination of that row of A_1^{-1} which corresponds to the column of A_1 in which R appears.

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New 108,000-kw. Generator for Grand Coulee. (*Civil Engineering*, Vol. 17, No. 10.)—A new 108,000-kw. generator as big as a six-room house has been completed by Westinghouse Electric Corp. for installation at Grand Coulee Dam, where it will increase generating capacity by the equivalent of 146,000 hp. This new giant and six similar machines already installed—the world's largest waterwheel generators—could supply power for all the industrial and residential requirements of a city twice the size of Pittsburgh.

Eventually the Grand Coulee program calls for 18 waterwheel generators, each of 108,000-kw. capacity. This tremendous concentration of energy will be used for pumping irrigation water to 1,200,000 acres of arid farm land in the Columbia Basin of central Washington and will set up a huge reservoir of electric power for industrial use in the West.

Completion of the 238-ton stator was the final step in the manufacturing job. Other parts of the generator, including a 72-ton steel shaft 32 ft. long, and a 464-ton rotor, already have been shipped. The generator is being assembled at Grand Coulee and is expected to go into operation early in 1948. Into its construction went approximately 750 tons of steel and more than 100 miles of copper wire.

R. H. O.

Airport Snow Removal. (*Heating and Ventilating*, Vol. 44, No. 10.)—A Pittsburgh engineer, John B. Sweeney, who has an annual battle with snow and ice at the airports in Pittsburgh under his control, has come up with a device designed to wipe out those menaces to year-round operation of airlines. The device operates like a gigantic flatiron, weighing 12 tons and literally ironing out the ice and snow completely on a 12-ft. swath of runway or roadway. It melts the snow and ice by means of two wide combustors which focus a blast of oil-fired, 2000 F. air on the area to be cleared.

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It travels at the rate of 10 to 15 miles an hour. It is mounted on an ordinary grader, and water that the great heat does not evaporate is wiped from the pavement by a large squeegee blade, working like an ordinary window cleaner. A relatively small amount of water is formed by the melting snow, since the water content of snow is only 2 per cent of its volume. When not in use the snow remover can be removed from the grader in a couple of hours and stored.

The 2000 F. blast is forced down to the runway through blowers operated by an air-cooled engine, the blast gaining its terrific heat from oil burners consuming 30 gal. an hour. Although they produce 8,000,000 Btu. per hour, the pavement is not damaged because of the speed of the machine and the fact that it is used in cold weather. Mr. Sweeney's machine will remove snow at an estimated cost of one cent a cubic yard. Furthermore, the machine makes a clean sweep of the area covered, leaving no icy spots to cause skids, and leaves no drifts to be cleared from the side of the road or runway, melting an extra 5-ft. strip on each side of its path.

The machine is the result of six years' work and experiment, with field test conducted last winter at the Greater Pittsburgh Airport. His first model was equipped with a two-way radio, which would be equally valuable on any kind of snow removal work, airports or roadways. Mr. Sweeney is Allegheny County Director of Aviation.

R. H. O.

A PHOTOMETRIC INVESTIGATION OF DUST.

BY

DOMINA EBERLE SPENCER [1] AND SELMA MALKIEL.[2]

1. INTRODUCTION.

One of the most troublesome effects of dust is the reduction in the amount of light through windows and skylights and from luminaires. The quantitative knowledge of the effect of dust is of practical importance in the design of luminaires and in the prediction of the performance of lighting systems.

A previous investigation [3] of the available data on the subject indicated a need for further experimental research. As a consequence, the present research was instigated with the purpose of determining how the transmittance of a dust film on a smooth surface varies with time and with angle of the surface. Measurements were made on nearly three hundred samples, which were exposed to the collection of dust in three locations. Measured transmittance of these dust films forms the basis for an empirical equation which gives the transmittance as a function of time and of angle.

2. EXPERIMENTAL PROCEDURE.

The dust was collected on plates of glass 5.1 × 5.1 cm. Cover glasses for Kodachrome slides were employed. They are easily obtained and are reasonably flat and uniform in thickness. The glass plates were held by phosphor bronze clips attached to steel frames. The frames were constructed of 6.4-mm. square steel rod, and the dust samples were held only along one edge so that the frames had little shielding effect.

The samples were carefully cleaned and were then mounted in pairs so that dust collected on one surface of each. The frames were constructed to hold the samples at the following angles: $\theta = 0$ (horizontal, facing up), $\pi/6$, $\pi/3$, $\pi/2$ (vertical), $2\pi/3$, $5\pi/6$, π (horizontal, facing down). The 0 and π plates were clamped together, likewise the $\pi/6$ and $5\pi/6$, the $\pi/3$ and $2\pi/3$, and two $\pi/2$ plates.

When dust had collected for a designated time, the samples were carefully removed and each was covered with a clean glass plate of the same size. The edge was then taped, leaving a dust film sealed between two glass plates. This sample could then be handled as much as desired without altering the film.

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[3] Parry Moon and D. E. Spencer, "Maintenance Factors," *Trans. Illum. Soc. Eng.*, Vol. 14, p. 211 (1946); and "Lighting Design," Addison-Wesley Press, Cambridge, Mass., 1948, Chap. VII.

Dust was collected in three locations. Thirty supporting frames were built, and fifteen were placed in a closed room at M. I. T. (Location B). This investigation was started on June 10, 1946. One set of samples was collected each week for four weeks. Thereafter, collections were made every four weeks. The last set of samples was collected on May 12, 1947, thus covering a period of 336 days.

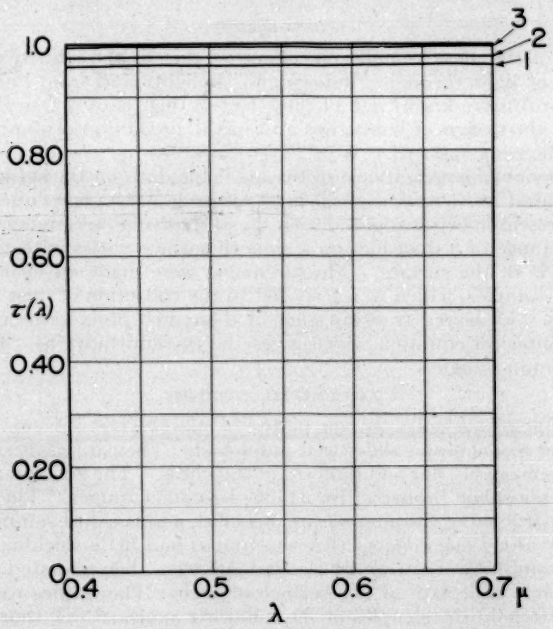

FIG. 1. Plot of transmittance τ of a dust film as a function of wavelength λ. Curves obtained by the Hardy photoelectric spectrophotometer.

1. $\theta = 0$
2. $\theta = \pi/3$
3. $\theta = 2\pi/3$.

The other fifteen sets of samples were assembled in a similar closed room at Tufts (Location C) on June 11, 1946 and collected on a corresponding schedule until May 13, 1947.

When about half of the collections of dust had been made and there were a number of vacant frames, these were put to use in a third location. Location A was Room 10–218 at M. I. T., a busy office in which the windows face a gravel parking lot and where dust is plentiful. A

set of glass plates was assembled in Location A on November 25, 1946. These were collected every four weeks until May 12, 1947 and provide data covering a period of 168 days.

Figure 1 shows curves of the transmittance of dust samples for Location B at 112 days. These spectral transmittance curves were obtained by means of the Hardy photoelectric spectrophotometer.[4] It is to be noted that the dust film is neutral in color.

A more direct method of measurement, which utilizes a bar photometer and a Lummer-Brodhun contrast photometer head, was finally decided on. Two 200-w. incandescent lamps of the same type were placed at the ends of the bench at a separation of 280 cm. A slide of clean glass was placed before the left lamp. The slide to be measured was placed before the right lamp. The instrument was calibrated by obtaining a visual balance with a clean glass slide also at the right end of the photometer. The same procedure was then repeated with a dust-film slide on the right. The transmittance of the dust film is

$$\tau = \left(\frac{l}{280 - l}\right)^2 \left(\frac{280 - l_s}{l_s}\right)^2, \qquad (1)$$

where τ = transmittance of dust film,
l_s = distance from photometer head to right lamp when clean slide is at right, cm., and
l = distance from photometer head to right lamp when slide containing dust film is at right, cm.

Five readings were obtained for each slide by balancing the bar photometer five times. The values of transmittance given in Tables I, II, and III are computed by averaging the five readings of l and substituting the average value into Eq. 1. An idea of the variation in the readings is given by the probable error. The probable error for a single reading was calculated for forty readings with eight slides collected from Location A after 140 days. The probable error was found to be 0.003 or about 0.3 per cent of the readings. Larger fluctuations in the data are consequently attributable to actual variations in the dust films.

If the values of transmittance given in the tables are compared with those obtained with the Hardy spectrophotometer, a discrepancy will be noticed. For instance, the value of τ in Fig. 1 for Location B at $t = 112$ days, $\theta = 0$, is 0.960; the corresponding value for the same slide measured on the bar photometer is 0.912. The explanation is that the Hardy spectrophotometer utilizes an integrating sphere and consequently records both scattered and directly transmitted light. The bar photometer, on the other hand, measures only the directly transmitted light and thus gives a lower reading.

[4] We wish to thank the Color Measurements Laboratory at M. I. T. for their kindness in obtaining the spectrophotometric curves of Fig. 1.

TABLE I.

Transmittance τ, Location A.

θ / t, days	0	π/6	π/3	π/2	π/2	2π/3	5π/6	π
28	0.941	0.961	0.971	1.000	1.000	1.000	1.000	1.000
56	0.908	0.925	0.935	0.983	0.984	0.990	0.992	0.990
84	0.837	0.879	0.908	0.986	0.988	0.986	0.986	0.986
112	0.816	0.837	0.897	0.984	0.985	0.987	0.988	0.988
140	0.790	0.839	0.885	0.986	0.987	0.994	0.993	0.993
168	0.787	0.801	0.864	0.978	0.983	0.993	0.984	0.992

TABLE II.

Transmittance τ, Location B.

θ / t, days	0	π/6	π/3	π/2	π/2	2π/3	5π/6	π
7	0.993	0.993	0.991	1.000	1.000	1.000	1.000	1.000
14	0.980	0.983	0.984	0.987	0.992	0.993	0.998	0.989
21	...	...	0.982	0.992	0.992	0.988	...	0.992
28	0.965	0.975	0.978	0.988	0.988	0.985	0.983	0.984
56	0.951	0.972	0.967	0.984	0.979	0.976	0.984	0.991
84	0.938	0.943	0.953	0.973	0.970	0.968	0.967	0.972
112	0.912	0.928	0.942	0.970	0.970	0.954	0.959	0.963
140	0.894	0.902	0.924	0.952	0.959	0.956	0.969	0.962
168	0.877	0.885	0.921	0.957	0.962	0.962	0.946	0.968
196	0.859	0.878	0.902	0.969	0.957	0.954	0.958	0.958
224	0.844	0.852	0.895	0.950	0.852	0.957	0.954	0.948
252	0.834	0.858	0.901	0.973	0.980	0.964	0.967	0.965
280	0.820	0.838	0.900	0.974	0.974	0.974	0.975	0.976
308	0.810	0.822	0.888	0.975	0.976	0.975	0.976	0.976
336	0.799	0.825	0.884	0.976	0.976	0.971	0.976	0.976

TABLE III.

Transmittance τ, Location C.

θ / t, days	0	π/6	π/3	π/2	π/2	2π/3	5π/6	π
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
14	0.985	1.000	1.000	1.000	1.000	1.000	1.000	1.000
21	0.958	0.982	0.992	0.993	1.000	1.000	1.000	1.000
28	0.973	0.968	0.982	0.982	0.982	0.987	0.982	0.983
56	0.980	0.979	0.988	0.997	0.997	1.000	1.000	0.998
84	0.965	0.979	0.986	0.989	0.986	0.992	1.000	1.000
112	0.932	0.910	0.988	0.951	0.951	0.996	0.946	0.948
140	0.922	0.946	0.952	0.992	0.992	0.984	0.992	0.994
168	0.899	0.918	0.940	0.983	0.978	0.984	0.980	0.982
196	0.893	0.906	0.940	0.996	0.993	0.996	0.994	0.990
224	0.850	0.875	0.913	0.982	0.982	0.980	0.985	0.981
252	0.845	0.871	0.919	0.984	0.989	0.993	0.982	0.992
280	0.818	0.828	0.873	0.987	0.991	0.960	0.995	0.973
308	0.813	0.837	0.890	0.973	0.977	0.988	0.990	0.994
336	0.790	0.830	0.866	0.965	0.968	0.949	0.965	0.968

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3. ANALYSIS OF RESULTS.

The experimental results are most conveniently used in the form of an equation. The equation

$$\tau = \delta + (1 - \delta)e^{-\mu t}, \tag{2}$$

where τ = transmittance of dust film,
t = time, days, and
δ, μ = constants,

was previously used to fit data on the collection of dust.[3] If dust collected uniformly with time, the transmittance would vary as $e^{-\mu t}$.

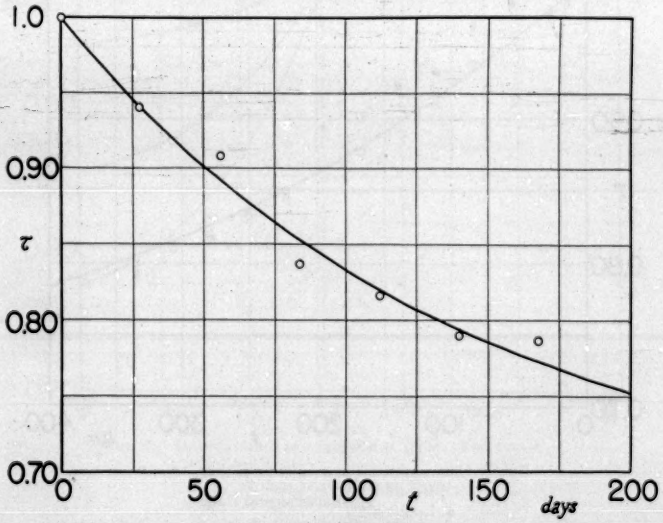


FIG. 2. Transmittance of a dust film as a function of time. Location A, $\theta = 0$.
○, Experimental values,
—, Calculated from Eq. 3.

This is nearly true for a clean surface, but as the dust becomes thicker, there is a tendency for as much dust to leave as arrives. A steady state is finally reached, at which $\tau = \delta$.

Figure 2 shows the experimental data for Location A, plotted as a function of time. The equation for the transmittance for a horizontal surface facing upward ($\theta = 0$) is

$$\tau = 0.681 + 0.319e^{-0.00740t} \tag{3}$$

and the resulting curve is shown in Fig. 2. The curves for other values

of θ were analyzed in a similar fashion, and it was found that all of them could be handled by using the same value of μ and different values of δ. The constant δ is, therefore, a function of θ. The relation may be represented by the equation

$$\begin{aligned} \delta &= 0.634 + 0.047e^{1.255\theta} \quad \text{if} \quad 0 \leqq \theta \leqq \pi/2, \\ \delta &= \delta(\pi/2) \qquad\qquad\quad \text{if} \quad \pi/2 \leqq \theta \leqq \pi, \end{aligned} \tag{4}$$

where θ = angle between the normal to the dust collecting surface and the upward direction (radian).

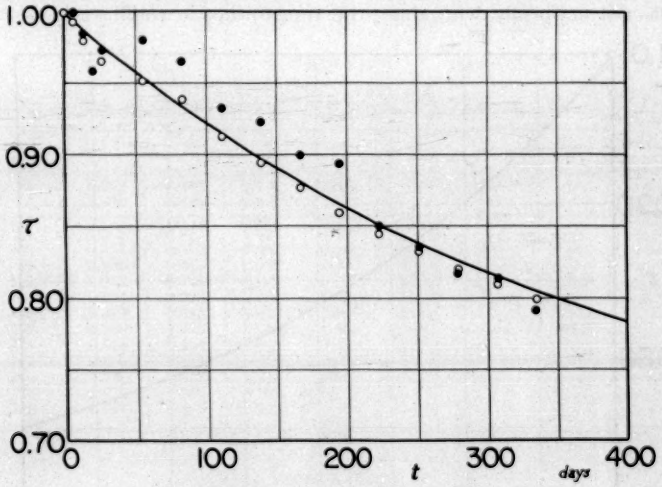


FIG. 3. Transmittance as a function of time. $\theta = 0$.
○, Location B } Experimental Data
●, Location C }
—, Calculated from Eq. 5.

Figure 3 shows data for Locations B and C for $\theta = 0$. The solid curve in Fig. 3 uses the same value of δ as for Fig. 2, but a different value of μ:

$$\tau = 0.681 + 0.319e^{-0.00250t}. \tag{5}$$

The equation for δ found with the data in Location A is found to be applicable to Locations B and C for all angles, but the new value of μ must be used (Eq. 5).

The data of the entire investigation can now be summarized in a single statement:

$$\tau(\theta, t) = \delta(\theta) + (1 - \delta(\theta))e^{-\mu t}, \tag{6}$$

where $\delta(\theta) = 0.634 + 0.047e^{1.258\theta}$ if $0 \leqq \theta \leqq \pi/2$,
$\delta(\theta) = \delta(\pi/2)$ if $\pi/2 \leqq \theta \leqq \pi$,
and $\mu = 0.00740$ in Location A,
$\mu = 0.00280$ in Locations B and C.

Figure 4 shows the transmittance as a function of θ for Location A and some of the data which were used to fit this curve. Note that the random variations are appreciable for surfaces facing downward, but

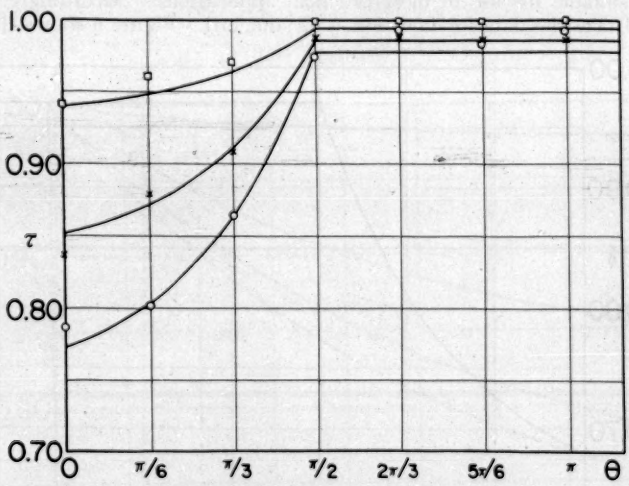


FIG. 4. Transmittance as a function of angle. Location A.
○, 168 days
×, 84 days } Experimental Data
□, 28 days
—, Calculated from Eq. 6.

these variations appear to follow no definite trend and are best replaced by a constant. Figure 5 is a plot of δ as a function of θ, calculated from Eq. 4. The physical significance of δ is simple: it is the ultimate transmittance for very large values of time.

An average [3] of previous data for collection of dust on windows was approximated by

$$\tau = 0.720 + 0.280e^{-0.0107t} \tag{7}$$

for a single horizontal film. The corresponding equation for Location A is

$$\tau = 0.681 + 0.319e^{-0.00740t} \tag{8}$$

and for Locations B and C,

$$\tau = 0.681 + 0.319e^{-0.00280t}. \tag{9}$$

Note that the ultimate values of τ are different by less than 6 per cent and the exponents are of the same order of magnitude. Considering the numerous uncontrolled variables of previous data, this agreement can be considered satisfactory.

It is suggested that the data on accumulation of dust in an office are suitable for use in most practical applications. Accordingly, we shall take the data for Location A as standard. Figure 6 shows these

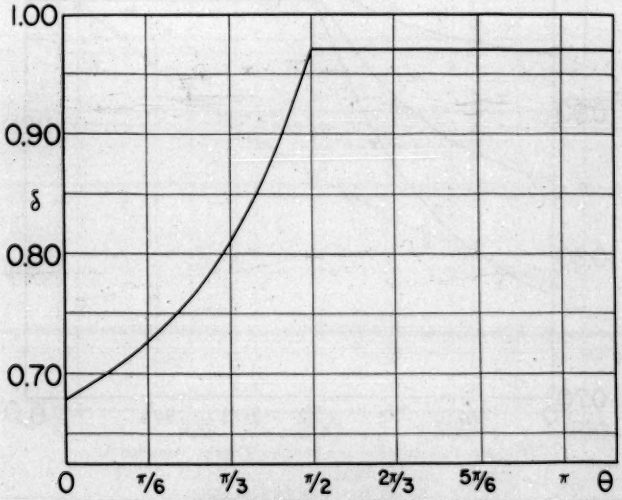


FIG. 5. δ as a function of angle. Calculated from Eq. 4.

data as represented by Eq. 6. The engineer should preferably obtain data for his locality. Lacking this specific information, however, he may find Fig. 6 and Eq. 6 helpful.

4. AVERAGE TRANSMITTANCE.

Equation 6 allows the calculation of transmittance for a dust film on any surface at any time t. In utilizing the results for the prediction of the behavior of luminaires, one needs also the average effect of dust that collects on a half cylinder or on a hemisphere. The former represents the upper half of a fluorescent lamp or a cylindrical enclosure, the latter represents half of an incandescent lamp or a diffusing globe.

The average transmittance τ_c for a half cylinder facing upward and with axis horizontal is obtained by substituting Eq. 6 for Location A in

$$\tau_c(t) = \frac{2}{\pi}\int_0^{\pi/2} \tau(\theta)d\theta$$

$$= 0.781 + 0.219\, e^{-0.00740t}. \qquad (10)$$

The average transmittance for a hemisphere facing upward is

$$\tau_s(t) = \int_0^{\pi/2} \sin\theta\tau(\theta)d\theta$$

$$= 0.817 + 0.183\, e^{-0.00740t}. \qquad (11)$$

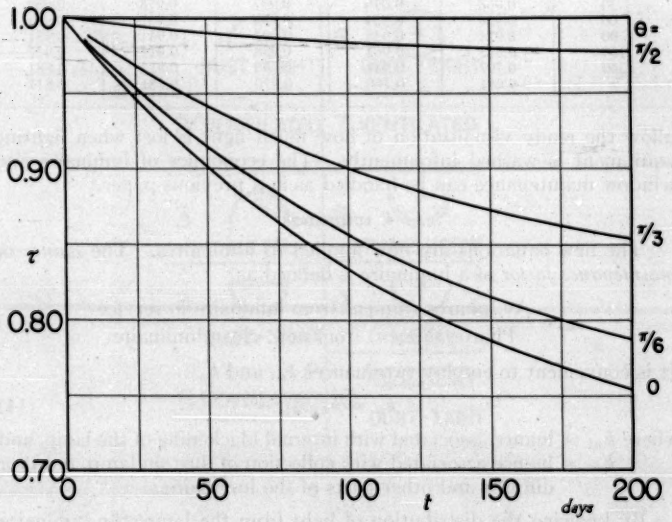


FIG. 6. Transmittance as a function of time (Location A). Calculated from Eq. 6.

The average values of transmittance for these surfaces facing downward are nearly $\tau(\pi/2)$, since the experimental data show that all angles from $\pi/2$ to π give the same transmittance.

All of the previous equations give τ as a function of time. These instantaneous values are not, however, the ones most useful in the treatment of luminaire maintenance. The average value maintained in practice is more useful than the instantaneous value. If the cleaning period for the luminaire is t_c days, then the average transmittance τ over

this period is

$$\tau = \frac{1}{t_c}\int_0^{t_c} \tau(t)dt$$

$$= \delta + \frac{(1-\delta)}{\mu t_c}(1 - e^{-\mu t_c}). \qquad (12)$$

Average values of τ computed from Eq. 12 for surfaces at angles of 0, $\pi/4$, $\pi/2$, and for τ_e and τ_s are given in Table IV. The tabulated values

TABLE IV.

Average Transmittance as a Function of Cleaning Time, t_c.

t_c, days	$\tau(0)$	$\tau(\pi/4)$	$\tau(\pi/2)$	τ_e	τ_s
30	0.968	0.976	0.997	0.978	0.982
60	0.938	0.954	0.994	0.957	0.964
90	0.914	0.935	0.992	0.941	0.951
150	0.873	0.904	0.988	0.912	0.927
360	0.792	0.844	0.980	0.857	0.881
∞	0.681	0.760	0.970	0.781	0.817

allow the ready visualization of how much light is lost when lighting equipment is washed infrequently. The economics of luminaire and window maintenance can be handled as in a previous paper.[3]

5. LUMINAIRES.

The new equations are now applied to luminaires. The *luance* or *maintenance factor* of a luminaire is defined as

$$k_m = \frac{\text{Av. pharos (lumen) from luminaire in service}}{\text{Pharos (lumen) from new, clean luminaire}}. \qquad (13)$$

It is convenient to employ two luances k_{m1} and k_{m2}:

$$k_m = k_{m1}k_{m2}, \qquad (14)$$

where k_{m1} = luance associated with internal blackening of the lamp, and
k_{m2} = luance associated with collection of dust on lamp, reflector diffusor and other parts of the luminaire.

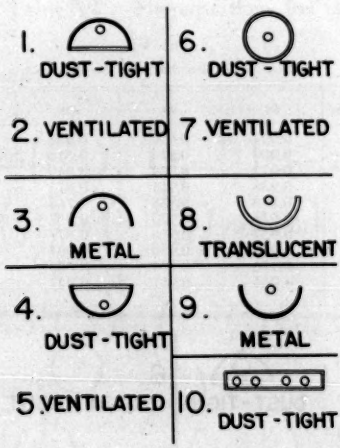
By knowing the distribution of light from the lamp, the luminaire designer can predict the effect of dust for any proposed luminaire. In this way, he can compare various designs, on the basis of maintenance. These calculations can be made to any desired degree of refinement for any specified luminaire, but usually it is sufficient to consider only a few representative types and to omit such refinements as interflections within the luminaire.

It is convenient to obtain a set of equations for luminaires containing fluorescent lamps and another for luminaires with incandescent lamps. Sketches of typical fluorescent types are shown in Table V. The equations follow the table.

TABLE V.

Luance k_{m2} of Fluorescent Luminaires.

Luminaire	t = 30 days	60	90	150	360	∞
1	0.997	0.994	0.992	0.988	0.980	0.970
2	0.948	0.905	0.870	0.810	0.700	0.564
3	0.985	0.970	0.959	0.939	0.902	0.853
4	0.968	0.938	0.914	0.873	0.792	0.681
5	0.932	0.871	0.826	0.748	0.612	0.454
6	0.988	0.976	0.966	0.950	0.918	0.875
7	0.964	0.928	0.902	0.856	0.771	0.664
8	0.974	0.952	0.933	0.902	0.840	0.758
9	0.966	0.934	0.910	0.867	0.788	0.686
10	0.982	0.966	0.958	0.931	0.886	0.826



Fluorescent Luminaires:

1. $k_{m2} = \tau(\pi/2)$

2. $k_{m2} = \tau(0)\tau^2(\pi/2)\left[\left(\frac{F_P}{F_L}\right) + \left(1 - \frac{F_P}{F_L}\right)\tau_c\tau(\pi/2)\right]$

3. $k_{m2} = \tau(\pi/2)\left[\left(\frac{F_P}{F_L}\right) + \left(1 - \frac{F_P}{F_L}\right)\tau_c\tau(\pi/2)\right]$

4. $k_{m2} = \tau(0)$

5. $k_{m2} = \tau_c\tau(0)\tau(\pi/2)\left[\left(\frac{F_P}{F_L}\right) + \left(1 - \frac{F_P}{F_L}\right)\tau_c\tau(\pi/2)\right]$

$$6.\quad k_{m2} = \left(\frac{F_P}{F_L}\right)\tau(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\tau_e$$

$$7.\quad k_{m2} = \tau_e\tau(\pi/2)\left[\left(\frac{F_P}{F_L}\right)\tau(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\tau_e\right]$$

$$8.\quad k_{m2} = \tau_e\left[\left(\frac{F_P}{F_L}\right)\tau^2(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\right]$$

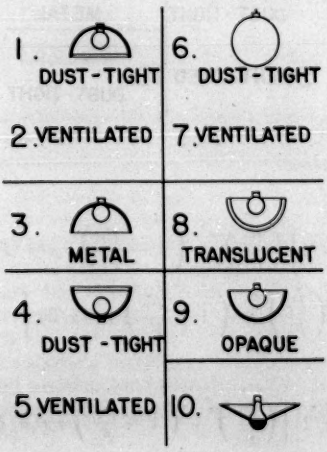
$$9.\quad k_{m2} = \tau_e\left[\left(\frac{F_P}{F_L}\right)\tau_e\tau(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\right]$$

$$10.\quad k_{m2} = \left(\frac{F_P}{F_L}\right)\tau(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\tau(0).$$

TABLE VI.

Luance k_{m2} of Incandescent Luminaires.

Luminaire	t_s = 30 days	60	90	150	360	∞
1	0.997	0.994	0.992	0.988	0.980	0.970
2	0.950	0.908	0.874	0.816	0.710	0.575
3	0.985	0.972	0.964	0.946	0.912	0.869
4	0.968	0.938	0.914	0.873	0.792	0.681
5	0.936	0.878	0.838	0.766	0.636	0.483
6	0.989	0.978	0.970	0.957	0.930	0.893
7	0.968	0.937	0.916	0.877	0.804	0.708
8	0.976	0.953	0.936	0.904	0.851	0.766
9	0.970	0.944	0.924	0.888	0.821	0.732
10	0.982	0.964	0.951	0.927	0.881	0.817



In Table V, Luminaires 1 and 2 are boxes with bottoms of diffusing glass or plastic. The reflector is opaque and the diffusing panel is usually flush with the ceiling. Number 3 consists of an opaque reflector with one or more fluorescent lamps. Numbers 4 and 5 are similar to 1 and 2 but are operated with the glass panel facing up. Numbers 6 and 7 are cylinders of glass or plastic surrounding fluorescent lamps. Numbers 8 and 9 are for ceiling lighting, the former being translucent while the latter is opaque. Number 10 is a new design consisting of a dust-tight box containing fluorescent lamps. The calculated values of luance k_{m2} are given in Table V on the assumption that $(F_P/F_L) = 0.50$ in all cases.

A corresponding set of results for luminaires using incandescent lamps is shown in Table VI. The equations for the luance of these luminaires are:

Incandescent Luminaires:

1. $k_{m2} = \tau(\pi/2)$
2. $k_{m2} = \tau(0)\tau^2(\pi/2)\left[\left(\frac{F_P}{F_L}\right) + \left(1 - \frac{F_P}{F_L}\right)\tau_s\tau(\pi/2)\right]$
3. $k_{m2} = \tau(\pi/2)\left[\left(\frac{F_P}{F_L}\right) + \left(1 - \frac{F_P}{F_L}\right)\tau_s\tau(\pi/2)\right]$
4. $k_{m2} = \tau(0)$
5. $k_{m2} = \tau_s\tau(0)\tau(\pi/2)\left[\left(\frac{F_P}{F_L}\right) + \left(1 - \frac{F_P}{F_L}\right)\tau_s\tau(\pi/2)\right]$
6. $k_{m2} = \left(\frac{F_P}{F_L}\right)\tau(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\tau_s$
7. $k_{m2} = \tau_s\tau(\pi/2)\left[\left(\frac{F_P}{F_L}\right)\tau(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\tau_s\right]$
8. $k_{m2} = \left(\frac{F_P}{F_L}\right)\tau(\pi/4)\tau^2(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\tau_s$
9. $k_{m2} = \tau_s\left[\left(\frac{F_P}{F_L}\right)\tau_s\tau(\pi/2) + \left(1 - \frac{F_P}{F_L}\right)\right]$
10. $k_{m2} = \tau_s$

The calculated values of Table VI assume $(F_P/F_L) = 0.50$, where F_P = pharos from lower half of lamp, F_L = pharos from entire lamp.

The first and second of the luminaires for incandescent lamps are light boxes with lens plates or diffusing glass for bottoms. The third is an ordinary metal reflector. Calculations are for a hemispherical

reflector but other shapes will give approximately the same results. Numbers 4 and 5 are similar to 1 and 2 but used to illuminate the ceiling. Numbers 6 and 7 are diffusing globes. Numbers 8 and 9 are translucent and opaque reflectors for ceiling lighting. Number 10 is a similar luminaire but with a silvered-bowl lamp.

Tables V and VI are similar to Table III of a previous paper.[3] In the earlier work, however, the effect of angle was not known and rough assumptions were made, based on the data available at that time. The tables of this paper make use of the new data on angle and the average values of Eqs. 10 and 11.

The new values of luance of luminaires are in fairly close agreement with those of the previous paper, differences usually running less than 5 per cent. In using these results it should be remembered that the formulae on which they are based are over-simplifications which neglect interflections. On the other hand, these approximations should be much more reliable than those generally used in industry at the present time.

6. SUMMARY.

The paper has presented data on the transmittance of dust films, as a function of time and angle. These results may be summarized by the equation

$$\tau = \delta + (1 - \delta)\, e^{-0.00740t},$$

where $\delta = 0.634 + 0.047\, e^{1.2350\theta}$ if $0 \leqq \theta \leqq \pi/2$,
$\delta = \delta(\pi/2)$ if $\pi/2 \leqq \theta \leqq \pi$,
and τ = transmittance of a single dust film (Location A),
t = time, days, and
θ = angle between the normal to the dust-collecting surface and the upward direction (radian).

These results were applied to the prediction of the effect of dust on the performance of various types of luminaires using fluorescent and incandescent lamps.

A MECHANICAL INTEGRAPH FOR THE NUMERICAL SOLUTION OF INTEGRAL EQUATIONS.

BY

PETER L. TEA, E.E., M.A.[1]

INTRODUCTION.

In a previous paper (1),[2] a graphical method was developed for the transformation of the integral of the product of two functions of y, given in Riemann form

$$I = \int^{b} \phi(y)K(y)dy, \qquad a \leqq y \leqq b, \tag{1}$$

into its equivalent Stieltjes form (2)

$$I = \int_{A}^{B} \phi(y)dY(y), \tag{2}$$

where

$$dY(y) = K(y)dy, \qquad Y(y) = \int_{0}^{y} K(y)dy, \tag{3}$$

$$A = Y(a), \qquad B = Y(b), \qquad Y(0) = 0.$$

The Y function was called the *functional line of* $K(y)$. It becomes the *functional surface*, Eq. 4, if it has a parameter x,

$$Y(y, x) = \int_{0}^{y} K(y, x)dy, \tag{4}$$

$$Y(0, x) \equiv 0, \qquad y = 0. \tag{5}$$

The functional surface contains at least one straight line element, the x axis. Planes perpendicular to the x axis cut the functional lines from the functional surface for the values of x cut by the planes.

1. *The Conversion of the Riemann Integral, Eq. 1, into Its Equivalent Stieltjes Form, Eq. 2, by Two Successive Projections of Either Function.*

Figure 1 shows the functional cylinder, Eq. 3, to the axes $0_1\phi$, $0_1Y(y)$, and 0_1y. The plane of $\phi(y)$ is placed parallel to the plane $\phi 0_1 y$, and in such manner that the origin 0 projects orthogonally from the plane of $\phi(y)$ to 0_1; and $\phi(y)$ projects to the functional cylinder then reprojects to the plane $\phi 0_2 Y(y) \perp 0y$. The second projection will be the Stieltjes curve corresponding to Eq. 2, the area under which between B and A

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

will be equal to the integrals, Eqs. 1 and 2, provided that the scales used in space in Fig. 1 are to the same unit value as that of the planimeter.

2. Shifting Functions for K.

The graphical method and that of the mechanical integraph are simplified considerably if K is of the form

$$K(y, x) = K(y \pm x); \tag{6}$$

then

$$Y(y \pm x) = \int_0^y K(y \pm x)dy = F(y \pm x) - F(0 \pm x), \tag{7}$$

for $y = 0$

$$Y(0 \pm x) \equiv 0, \tag{8}$$

for $x = 0$

$$Y(y \pm 0) = F(y \pm 0) - F(0 \pm 0). \tag{9}$$

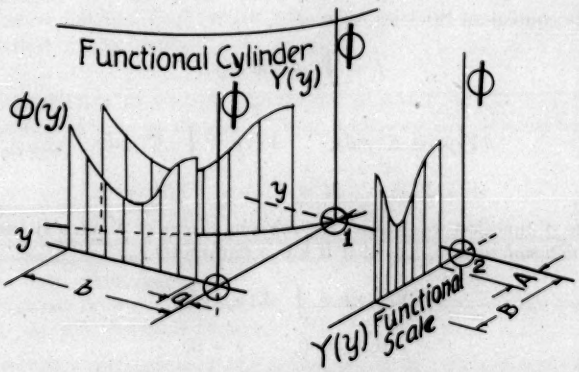

FIG. 1. Two orthogonal projections which convert the integral, Eq. 1, into its Stieltjes form, Eq. 2.

The K surface, Eq. 6, is a cylinder generated by the linear motion of the curve $K(y \pm 0)$ at 45° to the $+y$ axis if the sign of x is $-$, and 135° if the sign of x is $+$, and parallel to the xy plane. An example of such a surface is the symmetrical kernel, Eq. 18, Fig. 5.

The functional surface, Eq. 7, is not a cylinder, but it is composed of two cylinders: cylinder $F(y \pm x)$ at 45° or 135° to the y axis and parallel to the xy plane, like the K surface, and cylinder $F(0 \pm x)$ with elements parallel to the y axis. Cylinder $F(0 \pm x)$ merely displaces the functional lines vertically to pass them through the x axis, without changing their form. The functional surface, Eq. 19 of the kernel Fig. 5, is shown in Fig. 6.

If the functional cylinder $Y(y)$, Fig. 1, has as its base the functional

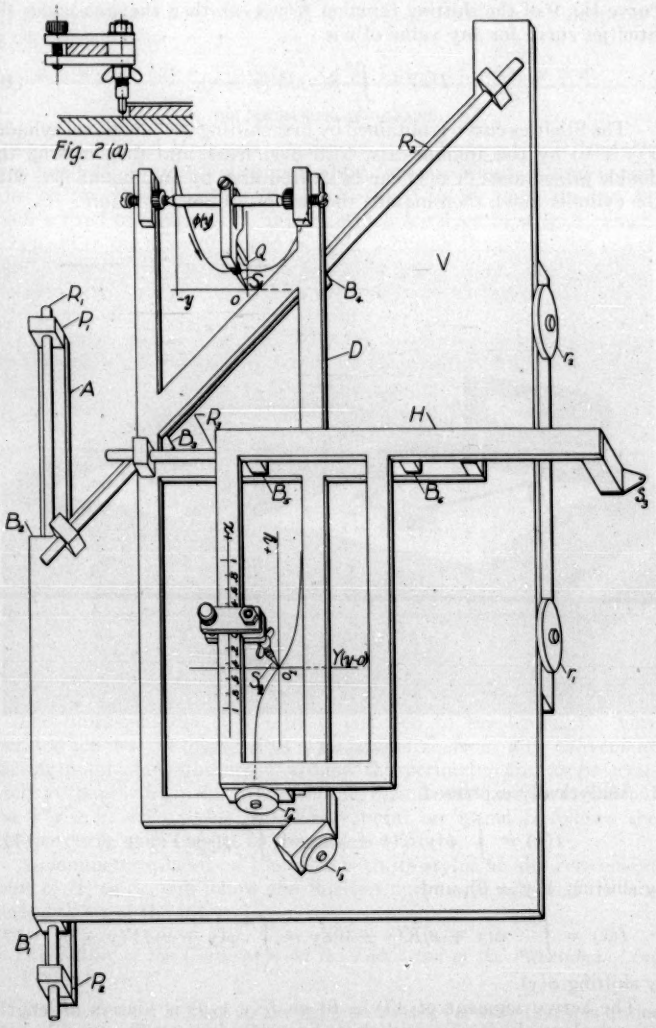

FIG. 2. Schematic drawing of the integraph.

curve Eq. 9 of the shifting function $K(y \pm x)$, then the area under the Stieltjes curve for any value of x is

$$I(x) = \int_A^B \phi(y) dY(y \pm x). \tag{10}$$

The Stieltjes curve is obtained by first shifting the functional cylinder $Y(y \pm 0)$ by the amount $\pm x$, with $\phi(y)$ fixed, and then making the double projections; or $\phi(y)$ can be shifted first, by an amount $\mp x$, with the cylinder fixed, then making the two projections as before.

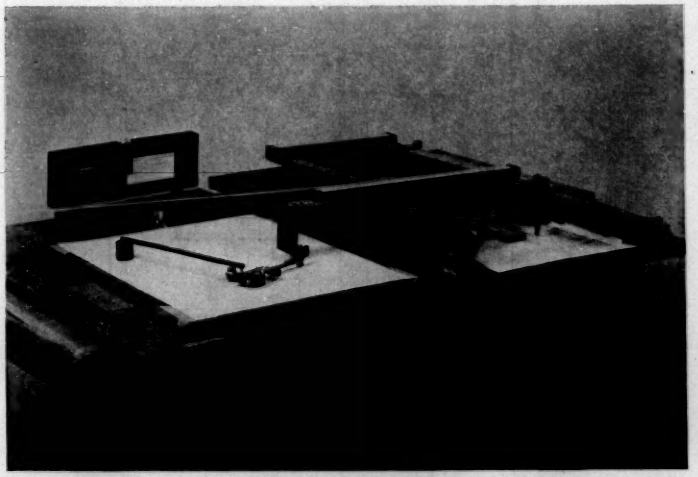

FIG. 3. Photograph of the integraph.

Analytically expressed

$$I(x) = \int_a^b \phi(y) K(y \pm x) dy = \int_A^B \phi(y) dY(y \pm x) \tag{11}$$

by shifting $Y(y \pm 0)$, and

$$I(x) = \int_{a \pm x}^{b \pm x} \phi(y \mp x) K(y \pm 0) dy = \int_A^B \phi(y \mp x) dY(y \pm 0) \tag{12}$$

by shifting $\phi(y)$.

The active segment of $Y(y \pm 0)$ or $K(y \pm 0)$ is always of length $(b - a)$ along the y axis, and shifts linearly with x. The total length of Y or K involved is $(b - a) + (d - c)$. The $(d - c)$ segment is

added to one side or the other of the $(b - a)$ segment for $x = 0$, depending on the sign of x.

$$a \leqq y \leqq b, \quad B = Y(b \pm x), \quad A = Y(a \pm x), \quad c \leqq x \leqq d.$$

I. THE MECHANICAL INTEGRAPH.

The mechanical integraph consists of three frames: V, D, and H, Figs. 2 and 3. Frame V is supported by two disc wheels r_1 and r_2 and slides on two brass bearings B_1 and B_2 (fixed to V) along the rod R_1 which is fixed to the posts P_1 and P_2, which are fixed to strip A, which is clamped to the drawing board or table. Frame D has one disc wheel r_3 and slides on two bearing B_3 and B_4 along the rod R_2 which is fixed by two posts to frame V. Rod R_2 permits diagonal motion of frame D with respect to V. Rod R_2 is accurately set at 45° to rod R_1. Stylus S_1 is mounted, by means of its holder, on a rod with cone pointed ends along which it can be fixed and with which it can rotate. The weight of the stylus holder and rod combination is supported by the cone bearings and by the rest Q. Rest Q is adjustable in length, and the stylus S_1 is adjustable in the distance of its point from the drawing board on which the $\phi(y)$ curve is fastened.

With a little practice, the two motions: V along R_1, and D along R_2 may be executed using a hand for each, so that stylus S_1 follows curve $\phi(y)$. The rods are lightly lubricated with clear Vaseline. They are drill rods, accurately ground and fairly straight. The slide bearings are of brass, reamed to an easy slide fit to the rods with no shake. Every point of frame D duplicates the motion of S_1 and describes the curve $\phi(y)$ with respect to the table.

Frame H has one disc wheel r_4 and slides on two bearings B_5 and B_6 along the rod R_3 which is fixed by means of two posts to frame D. Rod R_3 is accurately mounted at 90° to rod R_1 and at 45° to rod R_2, Fig. 2.

The functional line $Y(y \pm 0)$ is fastened to the frame V. One operator follows the curve $\phi(y)$ with stylus S_1 from any convenient starting point along the arrow around the perimeter and stops accurately at the starting point, while another operator follows the functional line $Y(y \pm x)$ with stylus S_2. Every point on frame H follows the Stieltjes curve with respect to the table.

A planimeter, placed on the table with its stylus at any convenient point of H, as S_3, will follow the Stieltjes curve and will read the numerical value of the integral.

1. The Setting of the Curve $\phi(y)$ on the Table and of the Functional Line $Y(y \pm 0)$ on V.

The curve $\phi(y)$ is fixed to the table so that its zero ordinate is under S_1 for motion of V only. With the holder of stylus S_2 set for $x = 0$, and with care not to move frame D, the functional line graph is set to frame

V with its zero ordinate line under S_2 for motion of H only. With the setting just described the planimeter reading will be the value of the Stieltjes integral for $x = 0$. For the integral for any value of x the holder of stylus S_2 is moved to that value; or with S_2 fixed at $x = 0$, $\phi(y)$ can be shifted to the x value.

If a shifting function is to have considerable use it is an advantage to cut its profile out of thin sheet iron on a band saw, and to file to the line by means of a jig file or by hand. The profile can be cut so that it can be used for a number of limit values of the integral. The stylus is replaced by a new piece with a No. 00 taper pin instead of the pointed stylus, Fig. 2(*a*). The sheet metal profile is fixed to a sheet of pressed wood $\frac{1}{8}$ in. thick by means of "Miracle Adhesive," and adjusted on frame V in the manner described. The taper pin is slightly raised from V, and rubs against the profile. A second operator is not necessary if the sheet metal profile is used. To provide a suitable force at the taper pin against the profile, adequate to actuate the planimeter without losing contact, a piece of cotton elastic band $\frac{1}{4}$ in. wide and about 3 ft. long was used with one end tied to frame H and the other end tied to the wall. Better ways can readily be devised, but the elastic band worked very well.

In case K is not a shifting function, then the curves for suitable x values can be drawn, all through the origin 0_1, and each is used in turn without changing the setting of stylus S_2, at $x = 0$.

2. Scales.

With rod R_2 at 45° to R_1, the scales of x and y must be equal. If the unit value of x and y is represented by m inches, and for the ordinate scale one unit of $\phi(y)$ is represented by n inches and one unit of the Y scale is l inches, then the area read on the planimeter is divided by nl. If different scales are desired for x and y the angle between R_2 and R_1 must be changed accordingly.

II. THE APPLICATION OF THE MECHANICAL INTEGRAPH TO THE EXPANSION OF FUNCTIONS IN A SERIES OF ORTHOGONAL FUNCTIONS.

A singly valued function $f(y)$ can be expanded in a series of orthogonal functions

$$f(y) = \sum_0^n a_q \varphi_q(y) \tag{13}$$

within an interval (a, b) if, *within the interval*,

$$\int_a^b \varphi_m(y)\varphi_n(y)dy = \begin{matrix} 0 & \text{for} & m \neq n, \\ 1 & \text{for} & m = n. \end{matrix} \tag{14}$$

The functions $\varphi_0(y)$, $\varphi_1(y)$ $\cdots$ are said to be orthogonal because all the integrals are zero for $m \neq n$, and are called normalized because all the integrals are equal to unity for $m = n$.

From Eqs. 13 and 14

$$a_q = \int_a^b f(y)\varphi_q(y)dy. \tag{15}$$

The expansion of single valued functions (or arbitrary curves) is greatly simplified by the use of orthogonal functions (3,4,5,6). The mechanical integraph may prove useful in boundary problems where the integration of Eq. 15 may offer difficulties. Functional lines of orthogonal sets of interest can readily be prepared.

III. THE APPLICATION OF THE MECHANICAL INTEGRAPH TO FREDHOLM'S INTEGRAL EQUATION OF THE FIRST KIND.

1. The Magnetic Field Along the Axis of Any Number of Co-axial Circular Loops Carrying a Current i_k, Fig. 4.

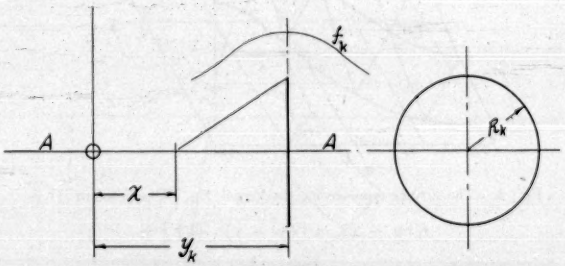

FIG. 4. Magnetic force f_k along the axis AA due to the current through a single turn at y_k.

The magnetic fields add vectorially at any point, and algebraically along the common axis AA,

$$F(x) = 2\pi\sum \frac{R_k^2 i_k}{[(y-x)^2 + R_k^2]^{\frac{3}{2}}}. \tag{16}$$

Helmholtz obtained a quasi constant magnetic field along the common axis between two equal concentrated coils carrying the same current and spaced a distance apart equal to the mean radius of the coils.

2. The Magnetic Field of a Cylindrical Solenoid of Radius 1 cm. Is to be Constant Along the Axis Between the Limits of the Solenoid.

The length of the solenoid is $2L$ and $\phi(y)$ represents the ampere turns per centimeter.

$$f(x) \equiv 2\pi \int_{-L}^{L} \frac{\phi(y)dy}{[(y-x)^2 + 1]^{\frac{3}{2}}} \equiv 2\pi c, \tag{17}$$

where $-L \leqq y \leqq L$ and, for this problem, $-L \leqq x \leqq L$.

This is a Fredholm integral equation of the first kind (3,4,5,7,15). The kernel is symmetrical in x and y, Eq. 18, and is shown pictorially in Fig. 5.

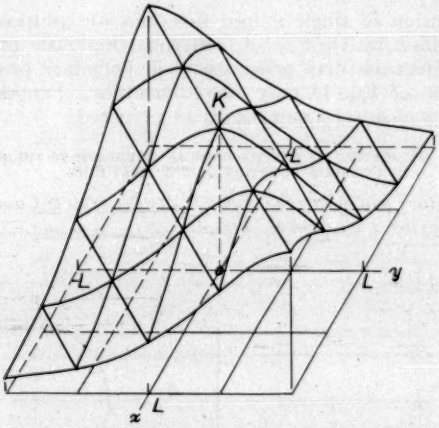

FIG. 5. The surface representing the kernel, Eq. 18, of problem III, 2.

$$K(y-x) = [(y-x)^2 + 1]^{-\frac{1}{2}}. \tag{18}$$

The functional surface

$$Y(y-x) = \frac{y-x}{\sqrt{(y-x)^2+1}} + \frac{x}{\sqrt{x^2+1}} \tag{19}$$

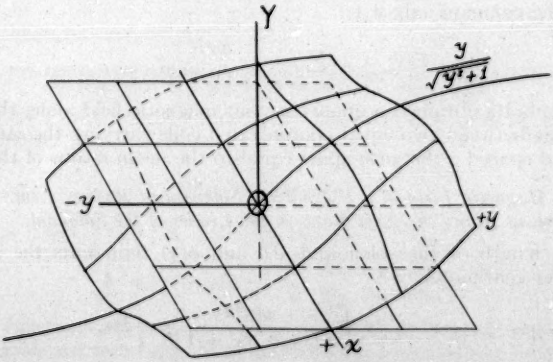

FIG. 6. The functional surface, Eq. 19, of the kernel, Fig. 5.

is shown in Fig. 6. It is generated by the functional line, Eq. 20, with the two motions discussed in the Introduction.

The functional line for $x = 0$, Eq. 20, was cut out of sheet iron of No. 26 gauge and fixed to frame V of the integraph; any value of Eq. 17 could be obtained for any curve $\phi_k(y)$ and any value of x.

$$Y(y - o) = \frac{y}{\sqrt{y^2 + 1}}. \tag{20}$$

The integral equation will be considered to be a set of $2n$ linear simultaneous algebraic equations

$$f(x_j) = 2\pi C = 2\pi \sum_{i=1}^{i=2n} K(y_i - x_j)\phi(y_i)\frac{L}{n}, \qquad j = 1, 2, \cdots 2n. \tag{21}$$

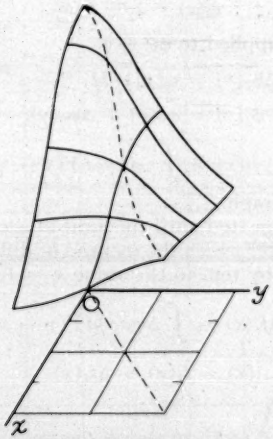

FIG. 7. The matrix of Eqs. 21.

The coefficients $K(y_i - x_j)$ for given x_j are given in a continuous form by the section of the kernel by the plane $x = x_j$.

The method used for solving Eqs. 21 was suggested by the successive approach to the solution of simultaneous linear equations having a large value of coefficients along the diagonal (8,9,10,11).

Since $\phi(y) = \phi(-y)$ the diagonal coefficient is

$$2\pi M = 2\pi[K(0) + K(2y)] = 2\pi[1 + K(2y)]. \tag{22}$$

Half the kernel obtained, by adding the pairs of coefficients for $\phi(y)$, is shown in Fig. 7. The problem was solved for $L = 1$, and Figs. 5, 6, 7 were drawn for the same limits.

3. Method of Successive Approach.

The initial value of the ampere turns function was taken to be

$$\phi_0(y) \equiv 1 \tag{23}$$

and, by integration,

$$f_0(x) = 2\pi\left[\frac{1-x}{\sqrt{(1-x)^2+1}} + \frac{1+x}{\sqrt{(1+x)^2+1}}\right]; \tag{24}$$

the value for $x = 0$ is

$$f_0(0) = 1.414 = c,$$

which is to be the value for $f(x)$ for all values of x.

The factor 2π was dropped since it cancels out in Eq. 26.

The error is

$$\epsilon_0(x) = c - f_0(x). \tag{25}$$

The correction to be applied to $\phi_0(y)$ is

$$\Delta\phi_0(y) = \epsilon_0(y)1/M, \tag{26}$$

$$\phi_1(y) = \phi_0(y) + \Delta\phi_0(y), \tag{27}$$

$$f_1(x) = \int_{-1}^{1} \phi_1(y)dY(y-x) \tag{28}$$

by means of the integraph.

The process was repeated until the limit of accuracy of the integraph was reached, at ϕ_{10}, after which the method of differences was used, with scale for ϕ multiplied by ten, to the value $k = 18$.

$$\Delta f_k(x) = \int \Delta\phi_k(y)dY(y-x), \tag{29}$$

$$f_{k+1}(x) = f_k(x) + \Delta f_k(x). \tag{30}$$

TABLE I.

Coefficients along the Diagonal.

x	0	0.1	0.2	0.3	0.4	0.5	0.55	0.6	0.7	0.8	0.85	0.9	0.95	1.00
M	2.000	1.943	1.801	1.629	1.477	1.345	1.305	1.263	1.196	1.148	1.130	1.114	1.101	1.089

It is seen from Fig. 7 that the diagonal coefficient is not the largest for x sections except near $x = 0$ and ± 1. However the application of Eqs. 25 to 30 shows a convergence, Fig. 8. The largest coefficient for x values becomes less dominant as x approaches ± 1. The successive approaches for $f_k(x)$ and $\phi_k(y)$ were re-calculated by proportion to anchor $f_k(\pm 1)$ at the value 1.414. Figure 8 indicated a slow convergence. Comparatively large plus and minus changes in ϕ have an almost cancelling effect on $f(x)$. The initial values of $\phi(y)$ show a rough

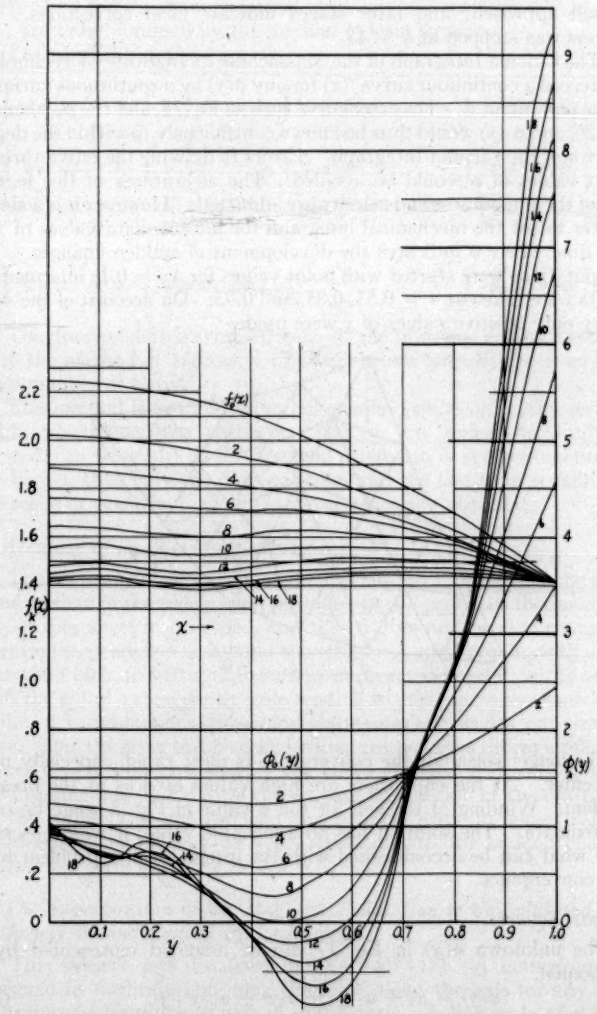

FIG. 8. Curves showing the gradual approach to the ampere turns function $\phi(y)$ as the magnetic force along the axis approaches $f(x) = 1.414$.

overall approach; and later stages indicate local corrections. The process was stopped at $k = 18$.

The Cinema Integraph of the Massachusetts Institute of Technology can record a continuous curve $f(x)$ for any $\phi(y)$ by a continuous variation of the parameter x. The error curve such as Eq. 25, and the $\phi_0(y)$ curve, Eq. 26, and $\phi_1(y)$ would thus be known continuously to within the degree of error of the Cinema Integraph. Errors in drawing the curve through point values of ϕ would be avoided. The advantages of this feature and of the smoothness and celerity are admitted. However it is a simple matter to set the mechanical integraph for intermediate values of x at any time where ϕ indicates the development of sudden changes. The computations were started with point values for $\Delta x = 0.1$; intermediate points were taken at $x = 0.55$, 0.85, and 0.95. On account of the symmetry only positive values of x were used.

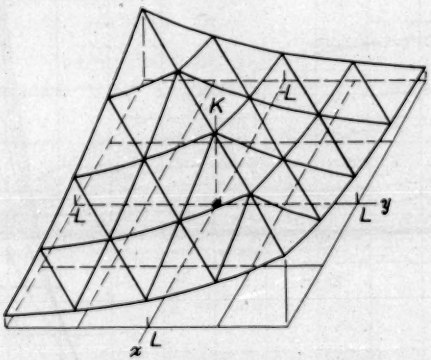

FIG. 9. The kernel in Buckley's interreflection problem.

For longer solenoids the convergence is more rapid, especially near the center. At the ends there are high values of ϕ as in the present problem. Winding of the coil for the ϕ value in Fig. 8 could be only approximate. The solution has no applicable value; it was chosen to show what can be accomplished with the integraph in a problem with slow convergence.

4. Least Squares.

The unknown $\phi(y)$ in Eq. 17 can be assumed represented by a polynomial,

$$\phi(y) = a_0 + a_1|y| + a_2y^2 + a_3|y^3| + \cdots. \tag{31}$$

Curves for 1, $|y|$, y^2, $\cdots$ etc., are used on the integraph, yielding

$$f(x) \equiv a_0\psi_0(x) + a_1\psi_1(x) + a_2\psi_2(x) \cdots, \tag{32}$$

where the ψ values are in graphical form, and the coefficients $a_0, a_1, \cdots$ etc., are to be obtained by the method of least squares.

The normal equations are

$$\left.\begin{aligned} a_0\int\psi_0^2dx &+ a_1\int\psi_1\psi_0dx + a_2\int\psi_2\psi_0dx + \cdots - \int f\psi_0dx = 0,\\ a_0\int\psi_0\psi_1dx &+ a_1\int\psi_1^2dx + a_2\int\psi_2\psi_1dx + \cdots - \int f\psi_1dx = 0,\\ &\cdots\\ a_0\int\psi_0\psi_ndx &+ a_1\int\psi_1\psi_ndx + \cdots - \int f\psi_ndx = 0. \end{aligned}\right\} \quad (33)$$

The discriminant is symmetrical. If the principal diagonal is dominant the method of successive approximations is practical, even for a large number of terms.

The method is not limited to polynomial expansion. Any series of single-valued functions, orthogonal sets, or any series graphically expressed, can be used. If the assumed expansion is in eigenfunctions of the kernel, then only the diagonal term and the last term remain, and the series of coefficients of the expansion is easily obtained.

5. Accuracy of the Mechanical Integraph.

The accuracy of operation of the mechanical integraph is about the same as that of the planimeter, provided that a profile of the functional line of thin sheet iron is used, and that ϕ does not develop changes of direction that are too many and too steep. A good profile, with offsets every half inch, to better than 0.01-in. accuracy is not difficult to attain, with the aid of a steel depth scale reading to 0.001 in., a mechanic's fine scribe, a watchmaker's glass, and draftsman's triangles and straight edge. But the error involved in drawing the ϕ curves through the point values depends on how sharply the curve changes in direction, where the intermediate points were chosen, and on the skill of the draftsman. In the problem of the solenoid the possible error in some places is estimated to be as high as 4 per cent, due to the sudden changes. Scales used were $m = 2$, $n = 2$, $l = 2$.

6. The Superposition of the Magnetic Fields Due to Co-axial and Uniformly Wound Coils of Finite Lengths.

This subject was discussed in "Vector" (12). A nomogram was designed to facilitate obtaining the force along the axis for any point of uniformly wound solenoids of any length. A line scale of suitable length can readily be made which would be more accurate than the nomogram. The reader is referred to the paper in "Vector" for the

design of long solenoids with concentric coils near the ends, to obtain a fairly uniform magnetic field along the axis for part of the length of the main solenoid.

IV. THE APPLICATION OF THE MECHANICAL INTEGRAPH TO FREDHOLM'S INTEGRAL EQUATION OF THE SECOND KIND.

For the integral equation of the second kind

$$\phi(x) = f(x) + \rho \int_a^b K(y, x)\phi(y)dy, \qquad (34)$$
$$a \leqq y \leqq b, \qquad a \leqq x \leqq b,$$

the successive substitutions

$$\phi_0(x) = f(x), \qquad \phi_{q+1}(x) = f(x) + \rho \int_a^b K(y, x)\phi_q(y)dy \qquad (35)$$

approach the solution as a limit, provided that there is convergence.

The mechanical integraph was used to solve Buckley's problem on the interreflection of light on the mat inner surface of a circular cylinder where $\phi_0(x)$ is the exciting luminosity with no reflection, that is, for $\rho = 0$, and the actual luminosity, with interreflection, is (13,16)

$$\phi(x) = \phi_0(x) + \frac{\rho}{2a} \int_{-l}^{l} e^{-|y-x|}\phi(y)dy. \qquad (36)$$

The exact kernel in Buckley's problem is given in Eq. 37. The kernel in Eq. 36, used by Buckley, is a close approximation to the exact kernel for the limits used. Both forms are shifting functions. The exponential form is convenient in applying Whittaker's method for the numerical solution of Eq. 34, by expanding the kernel in a series of exponentials by the method of Prony, thus effecting a separation of the variables (13,14,15).

$$K(y - x) = \frac{1}{2a}\left[1 - \frac{|y - x|(y - y^2 + 6a^2)}{(x - x^2 + 4a^2)^{\frac{1}{2}}}\right]. \qquad (37)$$

The values used were $\rho = 1$, cylinder radius $a = 1$, half length of cylinder $l = 1$. In Table II, the first three columns are comparable

TABLE II.

Comparative Values of $0(x)$ for Buckley's Problem Obtained by:

x	Mechanical Integraph	Graphical Method	Buckley Whittaker	Hedeman, Jr. Cinema Integraph
0.00	2.481	2.490	2.500	2.608
0.25	2.450	2.478	2.469	2.548
0.50	2.365	2.384	2.375	2.440
0.75	2.220	2.232	2.219	2.264
1.00	2.009	2.011	2.000	2.046

because the same approximate kernel of Eq. 36 was used; Hedeman, Jr. used the Cinema Integraph of M. I. T. on the exact kernel (16). Comparison of the results of the method of the mechanical integraph and the graphical method with the results of the Buckley-Whittaker method shows that the error is less than 1 per cent.

V. THE APPLICATION OF THE MECHANICAL INTEGRAPH TO THE CALCULATION OF THE RECIPROCAL KERNEL OF VOLTERRA.

The terms of Eq. 35 can be arranged in powers of ρ,

$$\phi(x) = f(x) + \int_a^b \Gamma(\rho^n; y, x) f(y) dy, \tag{38}$$

$$\Gamma = \sum_{n=1}^{\infty} \rho^n K_n(y, x), \tag{39}$$

where Γ is the reciprocal kernel of Volterra, and

$$K_1(y, x) = K(y, x), \qquad K_q(y, x) = \int_a^b K_1(s, x) K_{q-1}(y, s) ds \tag{40}$$

are the first and the qth iterated function of $K(y, x)$.

Equation 40, in its Stieltjes form, is

$$K_q(y, x) = \int_A^B K_{q-1}(y, s) dS(s, x), \tag{41}$$

where

$$dS(s, x) = K_1(s, x) ds, \qquad S(s, x) = \int_0^s K_1(s, x) ds. \tag{42}$$

The mechanical integraph can be used to calculate the numerical values of the iterated functions, Eq. 41, where x is still a parameter and y is now a parameter, with s as the variable which disappears on integrating.

Equation 41 is written as Eq. 43 in the form for a shifting kernel,

$$K_q(y, x) = \int_A^B K_{q-1}(y, s) dS(s - x). \tag{43}$$

The surface $K_{q-1}(y, s)$ is assumed known, and cut by the planes $y = y_k$, where $k = 1, 2, 3, \cdots$, equally spaced in the interval $(b - a)$, say, and the procedure of Eq. 11 or 12 is used, where s takes the place of y in Fig. 1.

If the kernel is symmetrical, then

$$K_{q-1}(y_k, x) \equiv K_{q-1}(y, x_k). \tag{44}$$

In Eq. 38 the functional line for $f(y)$ can be prepared and each term of the series can be integrated by the integraph, in the form of a series

of curves, functions of x, each term multiplied by the corresponding term of the power of ρ.

Improvements could be incorporated in a new model of the mechanical integraph. Ball bearings could be used, of either type: for rotation about a shaft, or translation along a shaft. The static and dynamic friction values are nearly equal, and less than for sleeve bearings if properly installed without strain. Sealed ball bearings are protected from dust and they contain all the lubricant that they will ever require.

To avoid parallax in using the pointed stylus, Coradi of Zurich, in his precision planimeters, uses "La Loupe Saphyr," consisting of a cylindrical piece of glass, axis vertical, height about $1\frac{1}{2}$ diameters, convex at the top, thus acting as a magnifier of 2:1. In the center of the flat bottom of the cylinder a thin circular disc of hard saphyr is inlaid which rests on and rubs on the paper. In the center of the saphyr disc there is a smaller circular spot which is made to ride the curve. This requires the operator to follow the curve with his eye directly over the cylinder. Van den Akker designed and built a mechanical integraph to integrate the product of two functions of y for a different purpose (17). He uses a small telescope, one for each graph, which projects a small spot of light on the curve. A pair of cross hairs could be used.

ACKNOWLEDGMENTS.

This research was helped materially by a small grant from the Research Corporation. Interesting suggestions were made by members of the Department of Physics at the City College, especially in regard to an optical method based on Fig. 1, which gives promise to develop into a method for obtaining a continuous curve $f(x)$ by moving either $\phi(y)$ or $Y(y \pm 0)$. My colleagues in the Department of Drafting offered valuable suggestions on the mechanical integraph which may be incorporated in a new design.

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Heat-Resistant Dyes. (*Heating and Ventilating*, Vol. 44, No. 10.)—The reaction to infra-red radiation of the dyestuffs used in clothing is an important factor in their warmth or coolness, according to new studies and experiments made at the Philadelphia Textile Institute as part of a research project sponsored by the Philadelphia Section, American Association of Textile Chemists and Colorists.

Experiments.—Wool, cotton, cotton duck, and rayon fabrics identical in every respect, including color, but dyed with different commercial dyestuffs, were used in the experiments conducted at the Philadelphia Textile Institute. Because of difference in the infra-red absorption characteristics of the dyestuffs used, the dyed fabrics, although identical in color, varied as much as 14 F. in temperature under prolonged exposure to sunlight.

Ice-Melting.—The heating effects on fabrics resulting from the varying infra-red characteristics of dyes is illustrated strikingly by the following experiment:

Two pieces of the same wool fabric are dyed black, one with Pontacyl Blue Black RC which is low in infra-red absorption and therefore reflects most of the infra-red in sunlight, and the other with Chromacyl Black W which absorbs most of the infra-red. The two fabrics are identical in appearance. Rectangular pieces of the two fabrics are laid over blocks of ice of the same size. Then they are placed side by side under infra-red lamps, which produce energy similar to that from the sun but at higher intensity and at a uniform rate. The experiment can be conducted equally well in sunlight, but it is slower and more troublesome because of varying sunlight intensity. Under the infra-red lamps, the ice melts very much faster under the fabric dyed with the high infra-red absorbing Chromacyl Black W than under the low infra-red absorbing Pontacyl Blue Black RC.

Burning.—Another interesting experiment also illustrated the heating effects resulting from infra-red absorption:

Two pieces of the same cotton poplin are dyed brown, one with Pontamine Brown BT, which reflects most of the infra-red and therefore is low in infra-red absorption, and the other with Ponsol Brown BB, which absorbs most of the infra-red. The samples are mounted side-by-side under high-intensity infra-red lamps. The fabric dyed with Ponsol Brown BB heats up rapidly, chars and then burns; while the fabric dyed with the low-absorbing Pontamine Brown BT merely becomes somewhat warmer.

In the leaflet entitled "How Cool is Your Color?" the institute points out that new discoveries concerning the wide variations in infra-red absorption characteristics of commercial dyestuffs "may prove of substantial significance to the textile and apparel industries and strongly influence the clothing habits of the American public."

R. H. O.

ADIABATIC FLOW OF HYDROGEN GAS THROUGH A ROCKET NOZZLE WITH AND WITHOUT COMPOSITION CHANGE.*

BY

S. S. PENNER[1] AND D. ALTMAN.[1]

ABSTRACT.

A procedure is described for determining the characteristics of adiabatic flow through a rocket nozzle with and without composition change. The method of calculation is illustrated for the expansion of pure hydrogen gas from a chamber temperature of 3506° K. and a pressure of 20.42 atm. to atmospheric pressure.

The study indicates that the exhaust velocity and temperature are highest for flow where complete equilibrium is reached at each temperature with respect to the reaction

$$H_2 \rightleftarrows 2H.$$

Flow with composition change requires a nozzle exit to nozzle throat area ratio somewhat greater than that determined for adiabatic flow without composition change for the same ratio of chamber pressure to exit pressure.

The residence time in a given temperature range is computed as a function of gas temperature for the two types of flow. The results of this calculation may be used to determine the minimum required reaction rates which allow composition changes during flow through the nozzle.

I. INTRODUCTION.

The performance of a rocket motor depends, among other factors, upon the nature of the flow process through the rocket nozzle. Performance calculations of rockets are generally carried out by assuming adiabatic flow without composition change (1).[2] Since reactions involving atoms or free radicals may be very rapid, it is possible that changes in chemical composition associated with atomic and free radical reactions do occur during flow through the rocket nozzle. Kinetics of homogeneous gas reactions in flow systems has been discussed in the literature (2). Problems concerned with the existence or nonexistence of equilibrium among internal-energy states are described elsewhere (3). Hydrodynamical considerations of flow through the nozzle are examined in a report from the Applied Mathematics Group of New York University (4).

The flow process through a rocket nozzle under conditions where complete chemical equilibrium is reached at each temperature is examined in the present paper. The results of this study may be tested for compatibility with the rates of atomic reactions if rate constants for the atomic reactions are available.

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

II. ANALYSIS OF EQUILIBRIUM AND CONSTANT-COMPOSITION FLOW PROCESSES.

A. Construction of Pressure-Temperature Plots.

A relation for calculating the pressure as a function of temperature during adiabatic flow through the rocket nozzle may be derived by use of the perfect gas law.

If the initial concentration of hydrogen molecules is 1 mole per cubic centimeter and α moles per cubic centimeter are dissociated at a given temperature, then the corresponding equilibrium constant K_p is given by the relation

$$K_p = \frac{(p_H)^2}{p_{H_2}} = \frac{4\alpha^2 p}{1 - \alpha^2}, \tag{1}$$

where p represents the total pressure and p_H and p_{H_2}, the partial pressures of atomic and molecular hydrogen, respectively. Differentiation of Eq. 1 leads to the result

$$d\left(\frac{K_p}{p}\right) = \frac{8\alpha d\alpha}{(1 - \alpha^2)^2}. \tag{2}$$

Since

$$d\left(\frac{K_p}{p}\right) = \frac{K_p}{p}\left(\frac{\Delta H}{RT^2}dT - d\ln p\right),$$

it follows that

$$\frac{K_p}{p}\left(\frac{\Delta H}{RT^2}dT - d\ln p\right) = \frac{8\alpha d\alpha}{(1 - \alpha^2)^2}, \tag{3}$$

where ΔH represents the heat required to dissociate 1 mole of hydrogen at the given temperature.

According to Eq. 1

$$\alpha = \frac{1}{2}\sqrt{\frac{K_p}{p}}\sqrt{1 - \alpha^2}.$$

Therefore, Eq. 3 may be rewritten as

$$\sqrt{\frac{K_p}{p}}\left(\frac{\Delta H}{RT^2}dT - d\ln p\right) = \frac{4d\alpha}{(1 - \alpha^2)^{\frac{1}{2}}}, \tag{4}$$

and if $\alpha \ll 1$

$$\sqrt{\frac{K_p}{p}}\left(\frac{\Delta H}{RT^2}dT - d\ln p\right) \simeq 4d\alpha. \tag{4a}$$

For adiabatic flow

$$(1 + \alpha)c_v dT + \Delta E d\alpha = -pdV, \tag{5}$$

where c_v is the heat capacity of 1 mole of gas at constant volume and ΔE represents the energy required to dissociate 1 mole of hydrogen.

From the perfect gas law, which is assumed to be valid,

$$pV = (1 + \alpha)RT$$

or

$$pdV + Vdp = (1 + \alpha)RdT + RTd\alpha. \tag{6}$$

Combining Eqs. 5 and 6 leads to

$$(1 + \alpha)c_v dT + \Delta E d\alpha = \frac{RT}{p}(1 + \alpha)dp - (1 + \alpha)RdT - RTd\alpha$$

or

$$c_v dT + RdT + \Delta E \frac{d\alpha}{1 + \alpha} + RT \frac{d\alpha}{1 + \alpha}$$

$$= c_p dT + \Delta H \frac{d\alpha}{1 + \alpha} = \frac{RT}{p} dp. \tag{7}$$

Solving Eq. 7 for $\frac{d\alpha}{1 + \alpha}$,

$$\frac{d\alpha}{1 + \alpha} = \frac{RT}{\Delta H} d \ln p - \frac{c_p}{\Delta H} dT. \tag{8}$$

If $\alpha \ll 1$,

$$d\alpha \simeq \frac{RT}{\Delta H} d \ln p - \frac{c_p}{\Delta H} dT. \tag{8a}$$

Introducing Eq. 8a into Eq. 4a leads to the result

$$\sqrt{\frac{K_p}{p}} \left(\frac{\Delta H}{RT^2} dT - d \ln p \right) \simeq \frac{4RT}{\Delta H} d \ln p - \frac{4c_p}{\Delta H} dT. \tag{9}$$

Equation 9 can be simplified to the relation

$$\left(c_p + \frac{B\Delta H}{RT^2} \right) dT = (RT + B) d \ln p, \tag{10}$$

where

$$B = \frac{\Delta H}{4} \sqrt{\frac{K_p}{p}}. \tag{11}$$

Since Eqs. 10 and 11 were derived on the assumption that $\alpha \ll 1$, they should not be applied at excessively high temperatures where the hydrogen molecules are extensively dissociated.

Equations 10 and 11 permit the construction of a pressure *versus* temperature plot through the rocket nozzle by successive approximation. Results obtained for the flow of 1 mole of hydrogen gas from a chamber temperature of 3506° K. at a pressure of 20.42 atm. to a pressure of 1 atm. are plotted in Fig. 1. Also shown in Fig. 1 is a similar pressure *versus* temperature plot for adiabatic flow without composition change. This curve was constructed (1,5) by use of Eq. 12:

$$\frac{d \ln p}{dT} = \frac{\gamma}{\gamma - 1} \cdot \frac{1}{T}. \tag{12}$$

The results shown in Fig. 1 indicate that the exit temperature for flow without composition change is 1650° K. The corresponding value for equilibrium flow is seen to be 2243° K.

B. Determination of $(1/p)(dp/dT)$ *as a Function of Temperature.*

The coefficients $(1/p)(dp/dT)$ may be determined as a function of temperature from Eqs. 10 and 12 for equilibrium and constant-composition flow, respectively. The calculated results for the flow of 1 mole of hydrogen from a chamber temperature of 3506° K. at a pressure of 20.42 atm. to atmospheric pressure are plotted in Fig. 2.

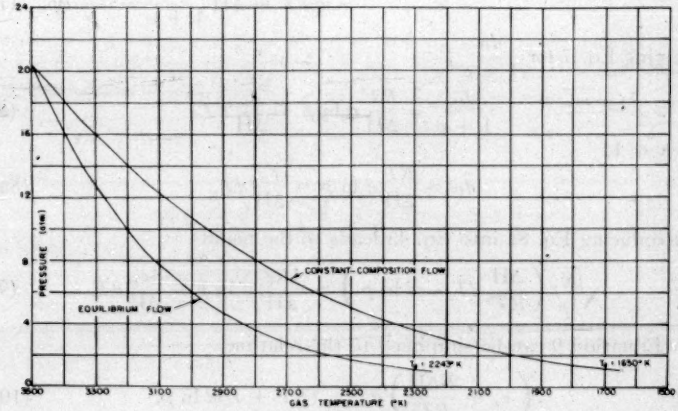

FIG. 1. Pressure-temperature history for the flow of hydrogen gas through a rocket nozzle (T_e = 3506° K., p_e = 20.42 atm.).

Reference to Fig. 2 indicates that $d \ln p/dT$ decreases continuously with increasing temperature for constant-composition flow but appears to increase for equilibrium flow for most of the temperature range from exit to chamber temperature. The coefficient $d \ln p/dT$ is seen to be appreciably greater for equilibrium flow than for constant-composition flow under corresponding conditions.

C. Determination of $-(1/v)(dv/dT)$ *as a Function of Temperature.*

The coefficient $-d \ln v/dT$, where v represents the linear flow velocity of the gases, can be calculated from the requirement of conservation of energy

$$Mvdv = -c_p dT - \Delta H \frac{d\alpha}{1+\alpha}. \qquad (13)$$

Equations 8 and 13 lead to the relation

$$Mvdv = -RT\frac{dp}{p}. \tag{14}$$

Assuming that $v = 0$ in the rocket chamber at $T = T_c$ (this is only a first approximation since the chamber is not infinitely wide), and integrating Eq. 14 from $v = 0$ at $T = T_c$ to $v = v$ at $T = T$, it is found that

$$\tfrac{1}{2}Mv^2 = \int_{T_c}^{T}\left(-RT\frac{d\ln p}{dT}\right)dT = \int_{T}^{T_c} RT\frac{d\ln p}{dT}\,dT. \tag{15}$$

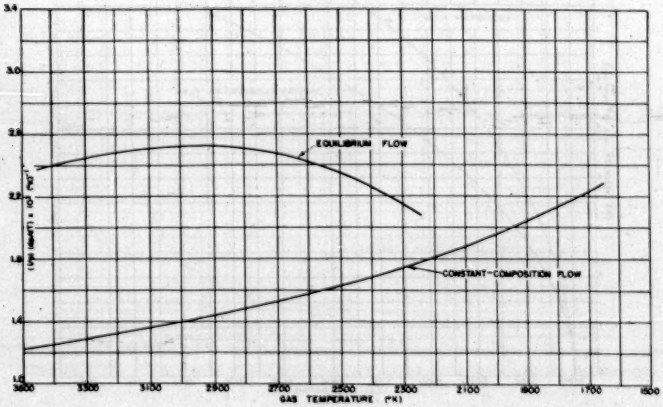

FIG. 2. The coefficient $(1/p)(dp/dT)$ as a function of temperature for the flow of hydrogen gas through a rocket nozzle (T_c = 3506° K., p_c = 20.42 atm.).

Dividing Eq. 14 by Eq. 15 leads to the result

$$-\frac{d\ln v}{dT} = \frac{RT\dfrac{d\ln p}{dT}}{2\displaystyle\int_{T}^{T_c}\left(RT\frac{d\ln p}{dT}\right)dT}. \tag{16}$$

For constant-composition flow through the rocket nozzle, Eqs. 15 and 16 should be replaced by Eqs. 17 and 18, respectively (1,5):

$$\tfrac{1}{2}Mv^2 = \int_{T}^{T_c} c_p dT \tag{17}$$

and

$$-\frac{d\ln v}{dT} = \frac{1}{2(T_c - T)}. \tag{18}$$

The gas velocities v and the coefficients $-(1/v)(dv/dT)$ as functions of gas temperature for equilibrium and constant-composition flow are plotted in Figs. 3 and 4, respectively. The exit velocity for equilibrium flow is seen to be 8.58×10^5 cm. per sec. compared with a value of 8.15×10^5 cm. per sec. for constant-composition flow. The higher exit velocity for equilibrium flow leads to the conclusion that better per-

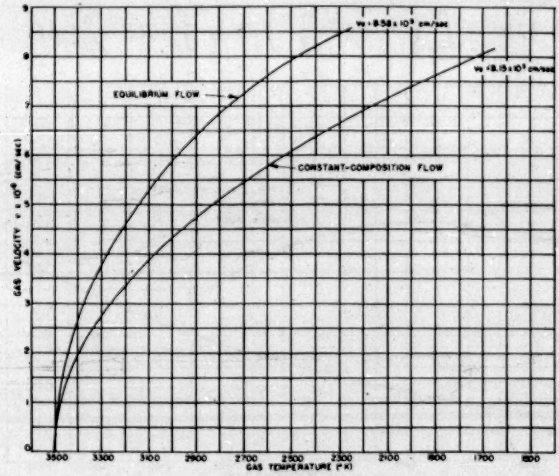

FIG. 3. Gas velocity as a function of temperature for the flow of hydrogen gas through a rocket nozzle (T_c = 3506° K., p_c = 20.42 atm.).

formance should be obtained with composition change than without composition change. The coefficients $-d \ln v/dT$ are seen to be of about the same magnitude for both types of flow at the same gas temperature.

D. Determination of the Nozzle Radius as a Function of Temperature.

For the present discussion, a cross section through the nozzle is assumed to be circular. The ratio of the nozzle radius at any point to the nozzle radius at the exit may, by application of the equation of continuity, be determined as a function of temperature.

Continuity of flow imposes the condition

$$\frac{d\rho}{\rho} + \frac{dv}{v} + \frac{dA}{A} = \frac{d\rho}{\rho} + \frac{dv}{v} + 2\frac{dr}{r} = 0, \tag{19}$$

where ρ represents the gas density, A is the cross-sectional area, and r is the radius of a cross section through the nozzle.

Since

$$\frac{d\rho}{\rho} = \frac{dp}{p} - \frac{dT}{T},^{*}$$

Eq. 19 may be written as

$$-2\frac{dr}{r} = \frac{dp}{p} - \frac{dT}{T} + \frac{dv}{v}$$

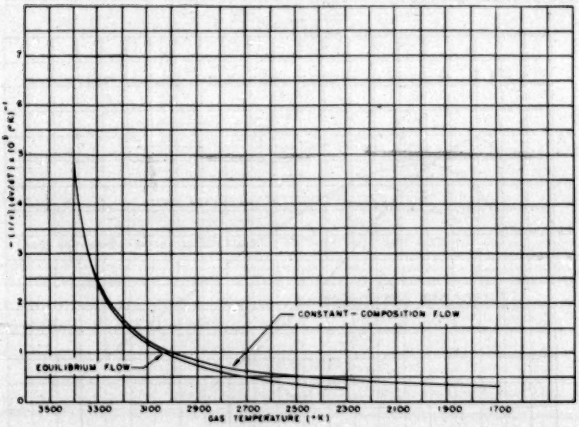

FIG. 4. The coefficient $-(1/v)(dv/dT)$ as a function of temperature for the flow of hydrogen gas through a rocket nozzle ($T_e = 3506°$ K., $p_e = 20.42$ atm.).

or

$$-2\frac{d\ln r}{dT} = \frac{d\ln p}{dT} - \left(-\frac{d\ln v}{dT}\right) - \frac{1}{T}, \tag{20}$$

and, therefore,

$$\ln\frac{r}{r_e} = \frac{1}{2}\int_T^{T_e}\left[\frac{d\ln p}{dT} - \left(-\frac{d\ln v}{dT}\right) - \frac{1}{T}\right]dT, \tag{21}$$

where the subscript e indicates conditions at the exit.

Graphical evaluation permits calculation of the ratio r/r_e as a function of temperature for equilibrium flow. The relation corresponding to Eq. 21 for constant-composition flow was derived previously (1,5) and is reproduced in the following equation:

$$\left(\frac{r}{r_e}\right)^2 = \left(\frac{T_e}{T}\right)^{\frac{1}{\gamma-1}}\sqrt{\frac{T_e - T_e}{T_e - T}}, \tag{22}$$

* Actually $d\rho/\rho$ should include a term in dM/M, that is, $\frac{d\rho}{\rho} = \frac{dp}{p} - \frac{dT}{T} + \frac{dM}{M}$. The contribution of dM/M to $d\rho/\rho$ is negligible in the present case. This is consisent with the assumption that $\alpha \ll 1$.

where γ represents the ratio of the specific heat at constant pressure to the specific heat at constant volume.

The ratios r_e/r are plotted in Fig. 5 as a function of temperature for the flow of 1 mole of hydrogen from $T_e = 3506°$ K. and $p_e = 20.42$ atm. to atmospheric pressure.

Reference to Fig. 5 indicates that the required area ratio $(r_e/r_t)^2$ for constant-composition flow is 3.55 while the corresponding value for equilibrium flow is 3.95.

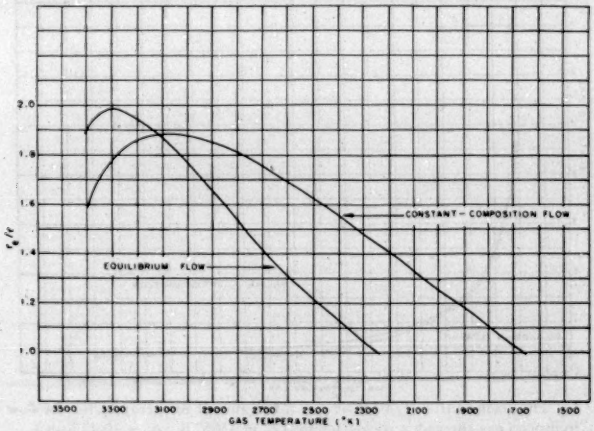

FIG. 5. The radius ratio r_e/r as a function of temperature for the flow of hydrogen gas through a rocket nozzle ($T_e = 3506°$ K., $p_e = 20.42$ atm.).

The coefficient $(1/r)(dr/dT)$ is equal to zero at the nozzle throat since the temperature decreases continuously from the chamber to the nozzle exit while $dr = 0$ at the nozzle throat. Accordingly, the gas temperature at the nozzle throat for equilibrium flow is determined by the condition

$$\frac{d \ln p}{dT} = \left(- \frac{d \ln v}{dT} \right) + \frac{1}{T}. \tag{23}$$

For constant-composition flow, the gas temperature at the nozzle throat is given by the relation (1,5)

$$T_t = \frac{2}{\bar{\gamma} + 1} \cdot T_e, \tag{24}$$

where $\bar{\gamma}$ represents an average heat capacity ratio during flow through the nozzle.

The gas temperature at the nozzle throat for constant-composition

flow is seen to be 3060° K. while the corresponding temperature for equilibrium flow is 3300° K.

E. Determination of the Time Spent in a Given Temperature Interval.

The coefficient $|-dt/dT|$ may be determined for equilibrium flow by following the procedure described for constant-composition flow (5).

If $|dx|$ is the linear distance measured parallel to the axis of the rocket nozzle, then

$$\left|-\frac{dt}{dT}\right| = \frac{1}{v}\frac{|dx|}{dT}. \tag{25}$$

A cross section through a rocket nozzle is shown in Fig. 6.

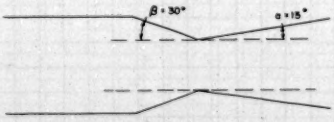

FIG. 6. Cross section through a rocket nozzle.

It is convenient to define $\tan\delta = \tan\beta$ between the nozzle throat and the chamber, and $\tan\delta = \tan\alpha$ between the nozzle throat and exit. It follows from these definitions that

$$dr = |dx|\tan\delta. \tag{26}$$

Replacing dr in Eq. 20 by its value from Eq. 26 leads to the relation

$$-\frac{2}{r}\left|\frac{dx}{dT}\right|\tan\delta = \frac{d\ln p}{dT} - \left(-\frac{d\ln v}{dT}\right) - \frac{1}{T}. \tag{27}$$

Combining Eqs. 25 and 27 leads to the result

$$\left|-\frac{dt}{dT}\right| = \frac{1}{v}\cdot\frac{r}{2\tan\delta}\left[\frac{d\ln p}{dT} - \left(-\frac{d\ln v}{dT}\right) - \frac{1}{T}\right]. \tag{28}$$

From Eq. 15, v is found to be

$$v = \sqrt{2\int_{T}^{T_e} RT\frac{d\ln p}{dt}\frac{}{M}\,dT}.$$

Therefore Eq. 28 becomes

$$\left|-\frac{dt}{dT}\right| = \left(\frac{r}{r_e}\right)\frac{r_e}{2\tan\delta}\left[\frac{d\ln p}{dT} - \left(-\frac{d\ln v}{dT}\right) - \frac{1}{T}\right] \cdot \sqrt{\frac{M}{2\int_{T}^{T_e}\left(RT\frac{d\ln p}{dT}\right)dT}}. \tag{29}$$

The expression for constant-composition flow, corresponding to Eq. 29, was previously shown to be (5)

$$\left|-\frac{dt}{dT}\right| = \left(\frac{r}{r_e}\right)\frac{r_e}{2\tan\delta}\sqrt{\frac{M}{2c_p T_e^3}} \times \left[\frac{1}{2\left(1-\frac{T}{T_e}\right)^{\frac{1}{2}}} - \frac{T_e}{(\gamma-1)T\sqrt{1-\frac{T}{T_e}}}\right]. \quad (30)$$

Calculated coefficients $|-dt/dT|$, obtained by use of Eqs. 29 and 30, are plotted in Fig. 7. Reference to Fig. 7 also indicates that $|-dt/dT|$

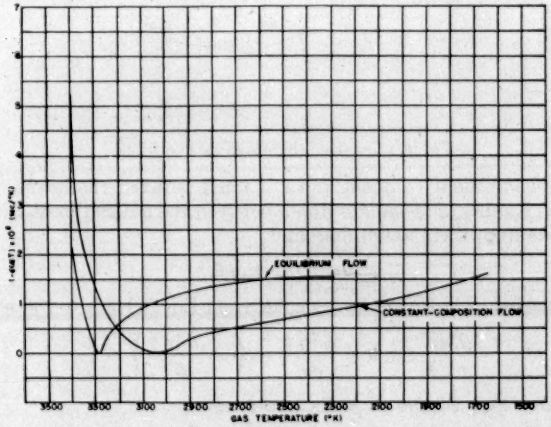

FIG. 7. The term $-dt/dT$ as a function of temperature for the flow of hydrogen gas through a rocket nozzle (T_e = 3506° K., p_e = 20.42 atm.).

between nozzle throat and nozzle exit is greater for equilibrium flow than for constant-composition flow. If this result is generally applicable, then the value of $|-dt/dT|$ for constant-composition flow may be advantageously employed for an estimate of minimum residence times of gases in given temperature intervals.

F. Determination of Residence Time of Gas Molecules in a Fixed Temperature Range.

The total time required for flow through the rocket nozzle cannot be determined accurately because of the approximation that the gas velocity in the chamber is zero.

The time spent in the temperature intervals between 3400° K. (which has been chosen arbitrarily) and any other temperature T° K. may be

evaluated by graphical integration of Eqs. 29 and 30 between $T°$ K. and 3400° K. The results of this calculation for constant-composition and equilibrium flow, respectively, are plotted in Fig. 8.

Reference to the data plotted in Fig. 8 indicates that 1.574×10^{-5} sec. are required during constant-composition flow before the gas is cooled from 3400° K. to the exit temperature of 1650° K. Similarly, 1.411×10^{-5} sec. are required during equilibrium flow before the gas is cooled from 3400° K. to the exit temperature of 2243° K. The time spent by the gases between the nozzle throat and the nozzle exit may

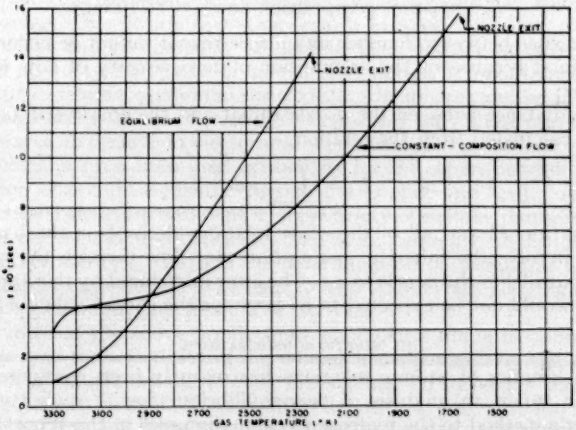

FIG. 8. Time spent in the temperature range between 3400° K. and $T°$ K. for the flow of hydrogen gas through a rocket nozzle (T_t = 3506° K., p_t = 20.42 atm., α = 15°, β = 30°, r_t = 5 cm.).

be read from Fig. 8 by subtracting the time spent between 3400° K. and the temperature at the nozzle throat from the time spent between 3400° K. and the exit temperature. This calculation leads to residence times between nozzle throat and nozzle exit of 1.31×10^{-5} sec. for equilibrium flow and 1.15×10^{-5} sec. for constant-composition flow.

The required residence times in fixed temperature intervals permit the determination of the minimum required reaction rates which allow composition changes during flow through the nozzle. This problem will be discussed in more detail in a subsequent report.

G. Distance Between Nozzle Throat and Nozzle Exit.

The linear distance between nozzle throat and nozzle exit may be determined for constant-composition and equilibrium flow, respectively,

for the chosen values of $r_e = 5$ cm. and $\alpha = 15°$. The ratios of r_e/r_t (Fig. 5) are required for this calculation. The distance L_t between nozzle throat and nozzle exit may be determined by use of the relation

$$L_t = \frac{r_e - r_t}{\tan \alpha}. \tag{31}$$

Calculations indicate that the conditions specified for the present analysis lead to the results

L_t (equilibrium flow) = 9.27 cm.
L_t (constant-composition flow) = 8.74 cm.

The distance between chamber and nozzle throat cannot be accurately determined because of the assumption of zero velocity of flow in the chamber.

The distance between the nozzle throat and the nozzle exit L_t may also be calculated from the relation

$$L_t = \int_{\text{throat}}^{\text{exit}} v\, dt. \tag{32}$$

Equation 32 and all of the other relations derived on the basis of Eq. 25 involve the inherent assumption that flow through the nozzle occurs parallel to the nozzle axis. The error introduced by this approximation would not be expected to be large for the chosen value of α.

III. KINETIC CONSIDERATIONS.

The kinetics of atomic reactions has recently been considered by Schaefer (6) in an analysis of near-equilibrium flow. Application of Schaefer's method to the hydrogen system discussed in this report leads to the conclusion that nearly complete chemical equilibrium is maintained during the adiabatic expansion of hydrogen through the rocket nozzle.

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NOTES FROM THE NATIONAL BUREAU OF STANDARDS.*

ELECTRONIC POWER SOURCE FOR ELECTROLYSIS.

A new constant voltage power supply for use in certain electrolytic determination and separation processes has been developed by the Electronic Instrumentation Laboratory of the National Bureau of Standards. This device, operating from a 110-volt A.C. line, supplies a stabilized and continuously adjustable D.C. voltage at current demands up to two amperes or more. Regulation is maintained with respect to a reference potential electrode, a principle recognized in recent years as a useful means of selectively depositing only those metals that it is desired to remove in electrolytic separation.

Electrolysis has been used extensively for the separation and determination of certain elements, particularly copper and lead. Another application consists in removing a large variety of metals from a solution so that the remaining constituents may be subsequently determined by some other method. Thus in the determination of aluminum in steel, the mercury cathode serves to remove iron and certain alloying constituents which might interfere in the analysis.

In most of the above applications no effort is made to control the voltage, and the selectivity of the process is determined by the type of solution in which the electrolysis takes place. Procedures using the more selective method of controlling deposition potential have been limited by the lack of suitable equipment to furnish and maintain the constant voltage required. Although several such devices have been built, they are not entirely suitable because of mechanical features which render their performance uncertain due to such factors as fouling of contacts, or because the electronic circuits used do not permit control under all conditions of operation.

The deficiencies of previous instruments are eliminated by the new power source developed by the Bureau. The voltage is regulated by a carefully engineered and compact electronic instrument that utilizes standard radio-type components. It is designed to produce a controlled output of several volts in loads as low as one ohm, though the same techniques could be used in apparatus built for higher outputs. In many applications, the conventional series-regulated power supply is at a disadvantage because of the high load currents which must be handled directly by vacuum tubes.

In the electrolytic separation application for which the instrument was designed, it is necessary to control the power supplied to a load, the

* Communicated by the Director.

plating electrode, with respect to the potential of a reference electrode. Too high a potential is indicative of excessive load power. Therefore, the function of the circuit is to reduce its output automatically until the control potential is at the desired value.

Internally, the instrument contains an amplitude-controlled oscillator, a power amplifier and rectifier, and two direct coupled stages of amplification. The oscillator, of the multivibrator type, operates at approximately 2,000 cycles per second, producing a variable amplitude square wave which is amplified by a pair of audio beam power tubes in push pull. These in turn are coupled through a step-down transformer to the low-voltage, high-current rectifier. An inductance and capacitance filter reduces ripple and noise to a very low value. This filtered voltage is the output of the instrument.

For regulation, a portion of the output voltage is compared to a voltage stabilized by a gaseous regulator tube. The difference between these voltages is amplified by the two direct coupled stages and used to control the amplitude of the oscillator signal. Connections are such that the oscillator amplitude is increased when the output voltage falls below the reference value. With the two stages of amplification, regulation is very precise, so that the output deviates only slightly from the reference voltage.

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The reference voltage is adjustable by a control in the panel of the instrument. This control, which therefore acts as the output voltage regulator, has a range of adjustment from zero to two volts, but higher voltages may be obtained by using only a part of the output for comparison. With a suitable voltage divider, a power supply capable of delivering up to 8 volts may be obtained.

As a result of the high degree of regulation, the instrument has a low effective internal impedance. This is about 0.005 ohm when the entire output voltage is used for comparison with the reference voltage. Hence the voltage drop from heavy load currents is small. The low internal impedance of the instrument makes it suitable as a battery substitute in those applications where constancy of output for load fluctuations is desired. The regulation ratio is approximately 0.5 per cent. In order to duplicate the performance of this instrument with a storage battery at an output of one volt, it would be necessary to employ a voltage divider with 200 amperes fixed drain.

The regulated voltage source may also be connected as a constant current source if desired. The voltage drop due to the passage of the load current through a selected fixed resistance then becomes the index which is compared to the reference potential. When applied in this manner, the instrument offers approximately the same advantages as when used for a voltage source.

One application in which the apparatus has been especially successful is the separation of copper from solutions preliminary to the polaro-

graphic determination of very small amounts of cadmium, lead, tin and other metals. The high degree of control attainable permits separation of elements whose half-wave potentials differ by only a few tenths of a volt. Additional experimentation is in progress in the Bureau's physical chemistry laboratory to determine the full scope of the instrument in electrochemical processes.

ELECTRODEPOSITION OF TUNGSTEN ALLOYS.

A method of producing satisfactory electrodeposits of tungsten alloys on metal surfaces has resulted from investigations conducted by Abner Brenner, Polly Burkhead, and Emma Seegmiller of the National Bureau of Standards. Smooth, thick deposits of tungsten alloyed with cobalt, nickel, or iron have been obtained from hot ammoniacal solutions containing organic acids. These alloys are quite hard and, what is more, retain their hardness at elevated temperatures, just as do the similarly constituted cast metallurgical alloys.[1]

The commercial importance of tungsten alloys is directly related to their use in applications—such as bearings, pistons, cylinders, dies, molds and machine tools—where hardness, corrosion resistance, or high strength at elevated temperatures are essential. Although efforts to electrodeposit pure tungsten have for many years proved unsuccessful, deposits of this metal alloyed with iron, nickel, and cobalt, have been produced by various processes. These methods, however, yielded specimens having unsatisfactory physical properties for many uses. Only a thin layer could be obtained, and this was weak because of cracks and oxide inclusions. The development of an improved process for depositing these alloys was therefore undertaken by the National Bureau of Standards.

Of the alloys investigated by the Bureau, cobalt-tungsten was found to deposit most easily. In some of its properties it resembles the alloy Stellite. A typical solution for depositing cobalt-tungsten contains:

Tungsten	25 g./l. (as sodium tungstate)
Cobalt	25 g./l. (as cobalt chloride or sulfate)
Rochelle salt	400 g./l.
Ammonium chloride	50 g./l.
Ammonia to a pH of 8.5 to 9	

Plating takes place at a temperature above 90° C. and at a current density of from 2 to 5 amperes per square decimeter. The Rochelle salt may be replaced by salts of other organic hydroxy acids. As anodes, either tungsten, the appropriate iron-group metal, or a tungsten

[1] For further technical details see a paper by Abner Brenner, Polly Burkhead, and Emma Seegmiller, "Electrodeposition of Tungsten Alloys Containing Iron, Nickel, and Cobalt," J. Research NBS, **39**, 351 (1947), RP 1834.

alloy may be employed. Tungsten anodes leave the least amount of residue when they go into solution, and may be used without bags. The solutions for depositing the iron- and nickel-tungsten alloys are very similar to that for cobalt-tungsten.

The maximum amounts of tungsten that may be obtained in this way are 35 per cent. in the nickel alloy, about 50 per cent. in the cobalt alloy, and about 60 per cent. in the iron alloy. However, for sound alloys of desirable properties, the tungsten content must be lower than the maximum. For example, the cobalt alloys should have not more than 30 per cent. of tungsten.

The nickel-tungsten and cobalt-tungsten deposits have good adhesion to steel and can be plated up to a thickness of 0.02 inch without becoming appreciably nodular. As formed, they are brittle and show a laminar structure under the microscope, but heating to suitable temperatures results in the disappearance of the laminations as the alloys become ductile.

The most interesting feature of the alloys obtained by this process is their hardness, which in the untreated nickel- and cobalt-tungsten alloys may be between 400 and 700 on the Vickers scale. The iron alloy is still harder, ranging from 700 to 900 Vickers; it is thus comparable in hardness to electrodeposited chromium.

Unlike most other electrodeposits, which soften with heat treatment, these alloys may increase 100 points or more in hardness upon heating to 600° C. The cobalt-tungsten alloy is of further interest in that it maintains a fair degree of hardness even at quite elevated temperatures. For example, at 700° C. the hardness of a carbon steel drops to about 50 Vickers, whereas the cobalt-tungsten alloy at this temperature has a hardness of over 290 Vickers.

The resistance of the nickel- and the cobalt-tungsten alloy to chemical attack is not much greater than that of either nickel or cobalt in the pure state, with the exception that they are much more resistant to nitric acid. In the salt-spray test, coatings of cobalt-tungsten alloy show less porosity and afford better protection to steel than do ordinary nickel coatings. However, the nickel-tungsten alloys are inferior in protective value to nickel coatings.

While tungsten alloys deposited from the solutions developed at the Bureau have not yet been utilized commercially, it is expected that they will find application on surfaces that require hardness and durability at elevated temperatures. Because of the excellent throwing power of the solutions, these coatings are more readily applied to irregular-shaped surfaces than is chromium plate. The electrodeposition section of the National Bureau of Standards will cooperate with those who are interested in plating small metal parts that can be subjected to a service test.

BORON-TREATED STEELS.

It is now recognized that the hardenability of some types of steel is materially improved by the addition of small amounts of boron. Apparently the effectiveness of the treatment with boron depends upon the steelmaking practice, and the amount and form of boron retained in the steels. The optimum effect on hardenability is obtained when boron is added in the form of simple or complex ferroalloys, commonly called "intensifiers," "special addition agents," or "needling agents," to thoroughly deoxidized heats in which the amount of boron recovered is within the range of about 0.001 to 0.005 per cent. Because one of the main roles of alloying elements in steels is to increase the depth of hardening or hardenability, this significant effect of boron suggested the possibility that it might be substituted for a part or all of the strategic elements commonly used as alloying agents in steels. During the war, to offset possible shortages of these critical elements, an extensive investigation was undertaken at the National Bureau of Standards to determine quantitatively the effects of boron on some properties of steels used for armor plate and other military applications.

The program was divided into two main parts; namely, (1) a study of the interrelationship between boron, carbon, and alloying constituents on some of the properties of steels made in the laboratory, and (2) a study of the properties of boron-treated steels of selected chemical composition made commercially under predetermined deoxidizing practices. Special attention was also directed toward the development of spectrographic and chemical methods for the accurate determination of small amounts of boron in steel. The investigation [1,2] was carried out at the National Bureau of Standards under the auspices of the War Metallurgy Committee, National Research Council, to which funds were transferred by the Office of Scientific Research and Development.

The work included a study of the cleanliness and structures of the steels as hot-rolled, normalized, and heat-treated. Determinations were made of the austenite and McQuaid-Ehn grain sizes, hardenability (Jominy), notch toughness (Charpy impact) at room and low temperatures, and tensile properties at room temperature. The investigation was extended with experimental steels to include a study of the effect of deoxidation practice and composition of ferroalloys containing boron, on the recovery of boron and its influence on the above properties; a determination of the recovery of boron on remelting in an induction

[1] For further technical details see "Influence of Boron on Some Properties of Experimental and Commercial Steel," by Thomas G. Digges and Fred M. Reinhart, *J. Research*, Nat. Bur. Standards, Vol. 39, p. 67 (1947) RP 1815. Reprints may be obtained from the Superintendent of Documents, Government Printing Office, Washington 25, D. C., at 30 cents each.

[2] The plans for the investigation were prepared jointly with a representative of the War Metallurgy Committee. The testing program for the commercial steels was prepared with, and approved by, the Subcommittee on Special Addition Agent Steels of the War Engineering Board, Iron and Steel Committee, and Army Ordnance.

furnace under both slightly and strongly deoxidizing conditions; and the determination of the effect of boron on the transformation temperatures and weldability of some of the experimental steels.

Approximately 250 experimental steels were made as "split" heats in an induction furnace and 20 commercial steels as a split heat in a basic open hearth furnace. Thus, a steel from each heat was prepared without boron (except in a few special cases) and used as a basis for determining the magnitude of the effect of boron on the mechanical properties.

Boron was determined chemically by a distillation-colorimetric method, using phosphoric acid and turmeric, with an accuracy of ± 0.0002 per cent of boron in the lower part of the range from 0.0005 to 0.006 per cent, and an accuracy of ± 0.0005 per cent in the upper part of this range. The average difference between determinations by chemical and spectrographic methods was about 0.0003 per cent of boron.

Initially normalized specimens, quenched from the usually recommended temperature range, were subjected to end-quench tests for hardenability (Jominy method). Although the relative hardenability of the various steels is obtained by a comparison of the Jominy curves, it is considerably more convenient to show the magnitude of the hardenability effect due to boron (or other variable) from heat to heat by comparing the distances from the quenched end of the Jominy bars corresponding to the same selected hardness values or structures. Comparisons, therefore, are made of the hardenability of the various steels on the basis of the distances from the quenched end of the bar for hardness values corresponding to both 50 and 95 per cent of martensite.

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Charpy impact tests for notch toughness were made with quenched and tempered specimens. Tests were carried out in duplicate at room and low temperatures in a Charpy machine of 224.1-ft-lb. capacity, with a striking velocity of the hammer of 16.85 ft-sec. The procedure for making the tests at low temperature consisted of cooling the specimens in an insulated bath containing equal parts of carbon tetrachloride and chloroform for a minimum of 30 min. at the desired temperature, and then quickly transferring to the impact machine and breaking. The desired temperatures were obtained by regulated additions of solid carbon dioxide, and as this material passes directly from the solid to a gas, no dilution occurred in the liquid bath. The temperature of the cooling bath was measured by means of a copper-constantan thermocouple and an indicating potentiometer. Rockwell "C" hardness was measured on the fractured specimens at room temperature.

Variations from nil to 0.006 per cent of boron additions made with either simple or complex intensifiers had no significant influence on the following properties of the steels: (1) cleanliness, except titanium or zirconium inclusions in some steels treated with complex intensifiers (nonmetallic inclusions), (2) hot working (experimental steels), (3)

transformation temperatures, (4) resistance to softening by tempering, (5) weldability (experimental steels), and (6) tensile properties of fully hardened and tempered specimens, except possibly an improvement in ductility when tempered at low temperatures.

Boron lowers the coarsening temperature of austenite. However, steels with relatively high additions of boron can be rendered fine-grained at heat-treating temperatures by the judicious use of grain-growth inhibitors such as aluminum, titanium, and zirconium.

The influence of boron on hardenability and on notch toughness (Charpy impact, V-notched specimens) of fully hardened and tempered steels varied with the base composition of the steels, the composition of the intensifiers, and the amount of boron present. The increase in hardenability due to boron was greater for basic open-hearth than for experimental steels prepared in an induction furnace. The notch toughness at room and low temperatures of the commercial steels fully hardened and tempered at high temperatures was also superior to that of the experimental steels of similar composition heat-treated alike.

The hardenability, as determined by the end-quench test, of many of the experimental and all the steels comprising a basic open-hearth heat were markedly improved by additions of boron. However, no definite correlation was found between the hardenability effect and the amounts of boron added or retained in the steels. In many of the experimental steels the optimum hardenability was obtained with small additions of boron (0.001 per cent or less retained), while in other steels the hardenability increased continuously with increase in boron. In other steels, the addition of boron as a simple or complex intensifier was either without effect or impaired the hardenability. In general, relatively small additions were more effective than large, and the complex intensifiers were more effective than the simple ones. The effectiveness of boron in enhancing the hardenability increased with the amounts (within limits) of manganese, chromium, and molybdenum. The hardenability of the boron-treated steels also varied with the state of deoxidation of the heat, and the final nitrogen content. High soluble nitrogen (and possibly oxygen) was detrimental to the boron effect on hardenability, but it was possible to retain the effect in high-nitrogen steels (low-soluble nitrogen) by fixing the nitrogen with strong nitride-forming elements such as titanium or zirconium.

For the commercial steels, the magnitude of the hardenability effect was independent of the amount of boron added or retained and the composition of the intensifiers.

The hardenability of some of the experimental and commercial steels treated with boron was affected by variation in quenching temperatures. In certain steels, the degree of hardenability was increased by increasing the quenching temperature above the usual recommended range, whereas in other steels the hardenability was not affected or was de-

creased by this change. The magnitude of the hardenability effect due to boron appears to depend upon the form in which it exists in austenite; not necessarily upon the total amount present.

The addition of small amounts of boron was often beneficial to the notch toughness at room temperature of the steels when fully hardened and tempered at low temperatures. When the steels were fully hardened and tempered at high temperatures the presence of boron, especially as relatively high additions with intensifiers containing titanium, was usually either without effect or was detrimental to notch toughness at room and subzero temperatures. Appreciable amounts of boron were retained in steel after remelting in an induction furnace under both normal and highly oxidizing conditions.

THE FRANKLIN INSTITUTE.

STATED MONTHLY MEETING, WEDNESDAY, APRIL 21, 1948.

The Stated Monthly Meeting of The Franklin Institute was held on April 21, 1948 in the Lecture Hall of the Main Building. The President, Mr. Richard T. Nalle, called the meeting to order at 8:15 P.M., with a capacity house in attendance.

The President stated that minutes of the monthly meeting for February had been printed in full in the March issue of the JOURNAL OF THE FRANKLIN INSTITUTE and if there were no corrections or additions the minutes would stand approved as printed. There was no comment.

The Secretary, Dr. Henry B. Allen, was then called upon for his report. The membership increase for the month of March was given as follows:

Active	36
Associate	23
Student	8

The total Institute membership as of March 31, 1948 was given as 5374.

The Secretary then announced the dinner to be given in honor of Mrs. Nellie Tayloe Ross, Director of the U. S. Mint. The dinner is scheduled to be held on April 29, and will be the opening of the exhibit of the new Franklin fifty-cent piece.

The President presented Dr. Hubert N. Alyea, Associate Professor of Chemistry at Princeton University, who spoke on "Chemistry of the Atom Bomb." He described, in a popular manner, the intense efforts of scientists to transmute one element into another, and how man's triumphant tapping of the energy locked up within the atom was accomplished. He also told how the Atom Bomb works and the peace-time uses to which Atomic Power may be put. Dr. Alyea illustrated with many chemical experiments throughout his lecture.

After a very interesting and edifying lecture the meeting was adjourned by a rising vote of thanks to the speaker of the evening.

HENRY B. ALLEN,
Secretary.

LIBRARY NOTES.

The Committee on Library desires to add to the collections any technical works that members would wish to contribute. Contributions will be gratefully acknowledged and placed in the library. Duplicates received will be transferred to other libraries as gifts of the donor.

Photostat Service. Photostat prints of any material in the collections can be supplied on request. Orders received in the morning are filled the same day. The average cost for a print 9×14 inches is thirty-five cents.

The Library and reading room are open on Mondays, Tuesdays, Fridays and Saturdays from 9 A.M. until 5 P.M., Wednesdays and Thursdays from 2 A.M. until 10 P.M.

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Christian Huygens and the Development of Science in the Seventeenth Century, by A. R. Bell. 220 pages, 14 × 22 cm., drawings and plates. New York, Longmans Green & Co.; London, Edward Arnold & Co., 1947. Price, $4.50.

Strength of Materials, by Joseph Marin. 464 pages, 16 × 24 cm., tables, drawings and illustrations. New York, Macmillan Co., 1948. Price, $4.75.

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Vacuum Tubes, by Karl R. Spangenberg. First edition, 860 pages, 16 × 23 cm., drawings and illustrations. New York, McGraw-Hill Book Co., Inc., 1948. Price, $7.50.

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Klystrons and Microwave Triodes, by Donald R. Hamilton and others. First edition, 533 pages, 16 × 23 cm., tables, drawings and illustrations. New York, McGraw-Hill Book Co., Inc., 1948. Price, $7.50.

BOOK REVIEWS.

HEAT, by A. G. Worthing and David Halliday. 522 pages, illustrations, 15 × 23 cm. New York, John Wiley & Sons, Inc., 1948. Price, $6.00.

This textbook covers a wide range of material, making it appropriate for an undergraduate course where introductory physics has been mastered, or for an advanced course of early graduate work, as well as for industrial reference.

The emphasis is on experimental methods. The inductive chapters evaluate the relative importance of the subject in question, then present the pertinent theories, experimental methods, results and problems. The thirteen chapters of this book cover laboratory procedure, temperature measurement, the expansions of liquids and solids, the dynamical theory of heat, calorimetry, specific tests of solids and liquids, thermal conduction, thermal properties of gases, thermodynamics, change of phase, heat engines, refrigerators, convection, and radiant energy.

H. N. MICHAEL.

FUNDAMENTALS IN CHEMICAL PROCESS CALCULATIONS, by Otto L. Kowalke. 158 pages, illustrations, diagrams, 15 × 22 cm. New York, The Macmillan Co., 1947. Price, $2.80.

This textbook is intended for use in the second year of a chemical engineering course. The process calculations discussed are to a large extent related to material balances.

The first portion of the book is made up of a discussion of the measurement of temperature, density and pressure. This is followed by an explanation of methods of expressing compositions of mixtures and solutions. After a study of the gas laws, the student is introduced

to material balances, and examples are given from manufacturing chemistry. Combustion calculations and heat losses make up the final chapters. Some commonly used tables are appended. The book is a careful exposition of the principles of industrial stoichiometry for the beginner. For this purpose it should be excellent.

G. S. GARDNER.

FIELD PRACTICE, by Elwyn E. Seelye. A Data Book For Civil Engineers. 306 pages, tables, drawings and illustrations, 13 × 21 cms. New York, John Wiley & Sons, Inc., 1947. Price $4.50.

Field engineering comprises extensive technological advances, many of which were developed quite recently. This book enables the inspector or field engineer to brief himself as to the essentials in the inspection and supervision of the proposed work. It is also useful as a field manual.

The contents of this volume, conveniently divided into sections on inspection and surveying, are very comprehensive.

The first part of the book deals with the inspection of concrete, masonry, structural steel, welding, bridges, painting, foundations on soil, pile driving, timber ropes and cables, soils, grinding, bituminous paving, pipe laying, and other miscellaneous items, even containing sample blanks of the pay-roll and expense records.

Inclusion of "Rules for Sampling," instructions for field tests, and other material, as well as check lists for work in concrete, bituminous paving, steel, welding and timber, adds to the field value of this book.

The second part reviews surveying and includes such items as construction stakeouts, circular curves, transition curves, vertical curves, earthwork computations, leveling, transit problems, and chapters on instrument adjustment and mapping.

General tables of trigonometric formulas, natural and logarithmic trigonometric functions, and other pertinent tabulated information round out this well-organized volume.

H. N. MICHAEL.

VERY HIGH-FREQUENCY TECHNIQUES, compiled by the staff of the Radio Research Laboratory, Harvard University. Volumes I and II. First Edition. 554 pages and 1057 pages, drawings and illustrations, 16 × 24 cms. New York, McGraw-Hill Book Co., 1947. Price $14.00 two volumes.

This work, in two volumes, represents a summary of the methods, theories, and circuits used by the Radio Research Laboratory, which will be of general interest to radio engineers and physicists. A compilation, drawn from many other sources besides the RRL, it does not pretend to be either a textbook or a comprehensive treatise covering all aspects of the v-h-f field. However, as a reference work, it should save considerable time and effort in the solution of problems in this field.

In its thirty-five chapters and over 1600 pages, the work covers a wide range of subjects.

A representative cross-section from the first volume would include ultra-high-frequency measurements, the various types of antennas, principles of direction finding, homing systems, power generation, and resnatron and magnetron operations.

The second volume includes such topics as filters, receivers, detectors, mixers, oscillators, and amplifiers.

In addition to the foot-noted bibliography the compilation contains a supplementary bibliography of recent articles.

H. N. MICHAEL.

NOMOGRAPHY, by Alexander S. Levens. 176 pages, illustrations, diagrams, plate, 15 × 23 cm. New York, John Wiley & Sons, Inc., 1948. Price, $3.00.

The material presented in this book provides a good working knowledge of the mathematical theory and design of nomograms. When the theory is mastered by the student, he is introduced to practical short-cuts which reduce the time required to design a chart. The lan-

guage of the text is clear and concise. The book itself is based on the material for nomography courses offered by the author at the University of Minnesota and the University of California. However, the practicing engineer who has retained a working knowledge of his algebra, plane geometry and logarithms will find no difficulty in using this book.

H. N. MICHAEL.

CHEMICAL PROCESS PRINCIPLES, by Olaf A. Hougen and Kenneth M. Watson. Volumes 1 to 3, 14 × 22 cm., illustrations. New York, John Wiley & Sons, Inc., 1947.

In the first edition of their work "Industrial Chemical Calculations" which appeared in 1931, the authors selected certain essential portions of physical chemistry and applied them to practical engineering calculations by direct and logical methods. The volume was so successful that it was followed by a second edition in 1936 in which the thermodynamic approach was much more strongly emphasized. The work now reappears in three volumes under a new name "Chemical Process Principles." The first volume covers applications of physical chemistry and the first law of thermodynamics, in much the same manner as in the second edition of the previous work, with much added material, and many new methods. In this volume material balances are discussed. The treatment has been elaborated to cover recycling, by-passing, changes of inventory and accumulation of inerts.

The second volume is devoted exclusively to thermodynamics. This represents a considerable elaboration as compared to the older work. The material is authoritative and carefully selected. The third volume covers a subject not previously considered in the second edition. The volume deals with homogeneous reactions, catalytic reactions, mass and heat transfer in catalytic beds, catalytic reactor design, and uncatalyzed heterogeneous reactions. The treatment is exceptionally fine. In the opinion of the reviewer "Chemical Process Principles" covers the field with a thoroughness and skill not duplicated by any other similar work.

G. S. GARDNER.

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ACTIONS OF RADIATIONS ON LIVING CELLS, by D. E. Lea. 402 pages, illustrations, plates, 15 × 22 cm. New York, The Macmillan Co., 1947. Price, $4.50.

This timely subject has been brought very much to the fore by revelations and developments in physics in recent years and the recent spur of research in certain directions in the medical field. It is a span between sciences which so often bring practical results of great value to mankind. This book reviews and summarizes present knowledge and gives pointed discussions by the author who has been engaged in studies of various biological effects of radiation over the past ten years. The radiations with which this book is concerned are the α-, β-, and γ-radiations of radioactive substances, X-rays, protons and neutrons which are grouped as ionizing radiations. Occasionally ultraviolet light is dealt with. Introductory chapters describe the physical properties and dosimetry of different radiations, and chemical effects of ionizing radiations to which is appended possible modes of biological action of radiation. Much of the rest of the book concerns the target theory. The biological effects of radiation to which this theory is applicable are those in which the effect studied is due to the production of ionization by the radiation in, or in the immediate vicinity of, some particular molecule or structure. Application of the target theory is referred to as the class in which the biological effect is believed due to a single ionization and there is interpreted in this manner the inactivation of viruses, the production of gene mutations and the killing of bacteria. A second class of action to which the target theory is applied is the production of certain chromosome aberrations in higher cells by radiations. The book reveals that considerable progress both on the experimental and theoretical sides of the applications of the theory has been made. Special attention is given to the subjects of the production of chromosome structural changes and the mechanism of induction of chromosome structural changes. An account is given of delay in division as an effect of radiation on a cell involving experimental materials such as bacteria, invertebrate eggs, and various rapidly dividing plant and animal tissues. An appendix contains supplementary calculations on the relation between dosage in air and energy dissipation in tissue, spatial distribution of ionization in irradiated tissue, and target theory calculations. There is a bibliography, author and subject index. The book is of value for all those whose work lies in this direction as a text and

reference. A great deal of effort has been put forth to make it usable by a wide circle of interested people, by arrangements for briefing prerequisite knowledge sufficient for a better understanding.

R. H. OPPERMANN.

CHEMICAL ENGINEERING FUNDAMENTALS, by Chalmer G. Kirkbride. 419 pages, illustrations, 15 × 24 cm. New York, McGraw-Hill Book Co., Inc., 1947. Price, $5.00.

The present volume is published as another work in the well known "Chemical Engineering Series." This series comprises many good books in the field of chemical technology. This textbook was written as an introductory work for the second year student in applied chemistry, applied physics, and applied mathematics and was introduced at the Agricultural and Mechanical College of Texas by the author, a former professor of the college. The text is presented in nine chapters. Chapter I is the introduction. Chapter II is on human relations (a matter not usually mentioned in chemical engineering texts). This chapter is well written and worthy of careful reading. Chapter III covers dimensional analysis and graphical presentations. Chapter IV acts as a refresher on some basic physical chemistry and thermodynamics, with a few pages on organic chemistry. Chapters V and VI take up, in a very thorough manner, the important question of material and energy balances. Chapters VII and VIII, two excellent chapters, are on static and dynamic equilibria, evidently a favorite subject of the author. Chapter IX, economic balance and finally Chapter X, presentation of technical results, show very clearly how large is the field covered by the book. The treatment is up to date, carefully written and reasonably accurate. In the opinion of the reviewer, this book is valuable for the student, engineer or chemist.

G. S. GARDNER.

A PRACTICAL EVALUATION OF RAILROAD MOTIVE POWER, by P. W. Kiefer. 65 pages, illustrations, 14 × 22 cm. New York, Steam Locomotive Research Institute, Inc., 1947. Price, $2.00.

Since the railroads today are faced with ever-increasing competition, they must try various ways to meet this rivalry. One of the more important methods is through the development of the ultimate in motive power. Mr. Kiefer, who is Chief Engineer of Motive Power and Rolling Stock of the New York Central System, offers an evaluation of present and proposed railroad motive power. Originally presented as a lecture at the Centenary Celebrations of the Institution of Mechanical Engineers in England, it has been made available in this form because of the present importance of the problem.

Mr. Kiefer begins his discussion with definitions of four fundamentals to be considered in evaluating motive power. These are (1) availability and its dependent counterpart, utilization, (2) over-all costs of ownership and usage, (3) capacity for work, (4) performance efficiency. He then proceeds to describe briefly the characteristics of the present types of motive-power units.

First is the reciprocating-type steam locomotive, still basic on American railroads. The need for fewer failures en route and a lessening of time spent in maintenance, servicing and inspection is commented upon. There have been notable improvements in this type over the past two decades and a table gives pertinent data for seven New York Central and one Pennsylvania Railroad engines during this period. Many of the features of the N.Y.C. "Niagara" 4-8-4 type are considered, this type being the culmination of the work to date on the New York Central. The kind of experimentation being carried on is evidenced by the fitting of one of the "Niagara" type with the Franklin system of poppet valve steam distribution. Exhaustive tests are being run to determine the relative merits of this new system in comparison with the more conventional system.

Of the new experimental types of locomotives mentioned there are the Pennsylvania Railroad coal-burning stoker-fired non-condensing steam-turbine locomotive introduced in 1944, a possible pulverized coal fired steam-turbine locomotive, and both coal and oil fired gas-turbine locomotives scheduled for delivery in 1949.

For electric locomotives the author notes their many operating advantages and the fewer

but more telling disadvantages, a lack of flexibility and a higher first cost and fixed charges for the necessary plant and equipment.

The Diesel-electric, which is the newest type in major use, has proved its value both in switching and road operation. Several different types are commented upon briefly.

The seventh chapter presents the heart of Mr. Kiefer's paper, in that he discusses the different types of motive power in relation to the fundamentals which he has previously defined. Availability of motive power has been the subject of several tests made on the New York Central of steam-engines and Diesels under comparable conditions of service, both passenger and freight. For costs evaluation there are offered in considerable detail actual and estimated annual operating costs for four different locomotives. In respect to capacity for work the author presents several charts showing power characteristic curves for various types of equipment. Performance efficiency is treated in two parts—over-all thermal efficiency at drawbar referred to fuel and over-all performance efficiency.

The concluding chapter presents a table summarizing the author's findings as to the relative evaluations of various motive-power types. It has been necessary to estimate the position of the gas-turbine and steam-turbine locomotives since few data are yet available. The tables indicate that on the majority of factors the straight electric is best, the Diesel electric is median and the reciprocating steam locomotive is last. The whole paper is a well-ordered presentation of the fundamentals to be considered in evaluating motive power and it forms a significant contribution to an understanding of the relative position of the types now in use.

GEORGE E. PETTENGILL.

THE CHEMISTRY OF PORTLAND CEMENT, by Robert Herman Bogue. 572 pages, illustrations, 15 × 24, cm. New York, Reinhold Publishing Corp., 1947. Price, $10.00.

The manufacture of portland cement is taken in this book as a distinct chemical industry. The treatment is principally chemical with side sketches of the mechanical phases where necessary to make a well rounded picture. The book is divided into three parts, the first being devoted to the chemistry of clinker formation, a section of ten chapters. This begins with a brief reference to the history and development of the cement industry, an outline of the manufacture of portland cement and proceeds through treatments on early experimentation on the constitution of portland cement, a discussion of the more important methods of investigations adaptable for research and for special examinations of portland cement and heat treatment in manufacture. The application of the petrographic microscope for examination of the principal compounds or phases of clinker, and the crystal analysis approach as accomplished by X-ray structure and character is given particular attention. The combination of cement components, the design and control of cement composition, and the calculation of the potential phase composition of the clinkers produced from raw mixtures of any known oxide composition.

Part two is devoted to the phase equilibria of clinker components. Following the development of the phase rule under the heading of the "Principles of High Temperature Phase Research," descriptions are given of the types of systems commonly encountered, the methods used for expressing the results of phase studies and the interpretation of phase diagrams. The technique of high temperature phase research is next considered which is followed by a detailed consideration of the characteristics of various systems including $CaO—Al_2O_3—SiO_2$, systems containing CaO and H_2O with Al_2O_3 and Fe_2O_3 are studied and this is followed by treatments on the aluminate and ferrite complex salts, the heat of hydration, the structure of hydrated cements, the various aspects of the settling process, and the progress of the reactions in the set cement by virtue of which the paste acquires the characteristics necessary for the development of strength and durability in concrete structures. A final chapter is devoted to tests for cement quality. There is included a subject and author index.

The work is a comphehensive and thorough treatment developing modern concepts based on an historical background.

R. H. OPPERMANN.

NOTES FROM THE BIOCHEMICAL RESEARCH FOUNDATION.

ON ENZYMES IN TUMORS.

By Ernst Waldschmidt-Leitz, Ellice McDonald and Co-workers.

(This paper "On Enzymes in Cancer" was published fifteen years ago, in Hoppe-Seyler's *Zeitschrift für physiologische Chemie*, **219**: 115–127 (1933). It attracted little attention for two reasons: it was unfortunately in German and few people were then interested in enzymes, much less enzymes in cancer. Now enzymes in cancer are much studied and this translation is published in order to make this early information available to others.

Ellice McDonald.)

The results of O. Warburg's researches, by means of which the peculiarity of malignant tumors was recognized to be a metabolic problem, give the task to cancer research of examining the enzyme composition of tumors. The enzyme systems determine the direction and extent of metabolic changes, and must be described in the course of tumor development in order to follow the exchanges taking place in the tumor animal; to trace, for example, the relation between the enzyme picture in the normal and in the cancerous organs. This problem has as yet been studied but little. No researches of this kind have included a systematic series of functionally important intracellular enzymes.

The present work, which examines the proteolytic systems of organs and of experimental tumors both sarcoma and carcinoma, includes cathepsin, peptidases, arginase, desamidases, phosphatase and catalase. Pathological analyses showing the proportion of different structural elements during the development and aging of the tumors serve as a foundation for the enzymatic research. We distinguish between parenchymatous, fibrous and necrotic portions. The enzyme analysis is, therefore, referred to the structural composition of the tumor, although the figures used for the latter were necessarily based rather on estimations than exact numerical measurements. The results so obtained, however, are uniform enough to show the main trend of alterations in the concentration of single enzymes establishing a definite relation with the structural elements of the tumors; thus at the same time they confirm the authenticity of the pathological findings.

Most of the enzymes studied, namely cathepsin, phosphatase and desamidases, show a steady and pronounced decrease in the enzyme concentration (see Figs. 1 to 4) with the aging of the tumor and the consequent increase in necrotic tissue. It is to be concluded that these

enzymes are confined almost exclusively to the parenchyma of the tumors. Arginase, on the other hand, shows the opposite picture; the amount increases greatly with increasing necrosis (Fig. 5). It appears that arginase is to be found predominantly in necrotic tissue substance

FIG. 1. Cathepsin content and tumor aging.

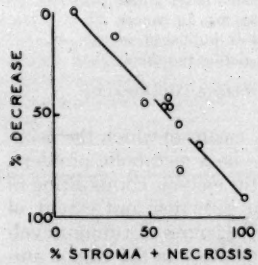
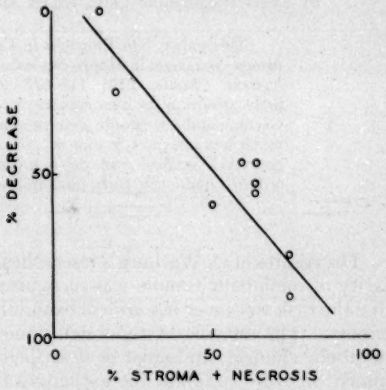


FIG. 2. Phosphatase content and tumor aging.



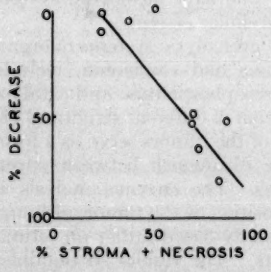


FIG. 3. Adenylic acid desamidase content and tumor aging.

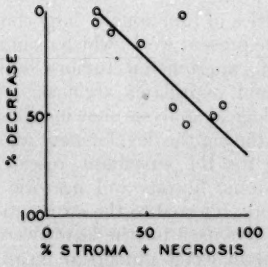


FIG. 4. Guanine desamidase content and tumor aging.

and only in small amounts in growing tissue. An explanation for the rise in arginase, particularly in decomposition tissue, may be sought by the assumption that the action of this enzyme is especially necessary in the later stages of cell decomposition, since, according to Krebs [1] it

[1] H. A. KREBS, Hoppe-Seyler's *Zeitschrift für physiologische Chemie*, 210: 33 (1932).

plays an important part in the formation of urea. The amount of the peptide-splitting enzymes, on the other hand, amino-polypeptidase and

FIG. 5. Arginase content and tumor aging.

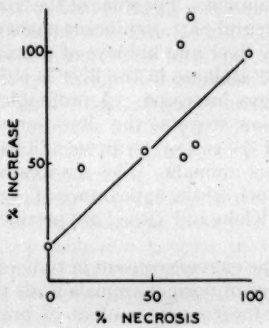
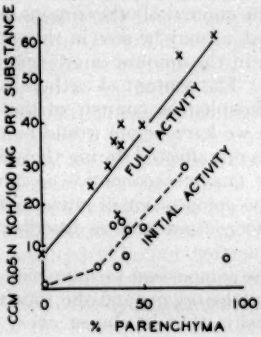


FIG. 6. Degree of activation of cathepsin and tumor aging.



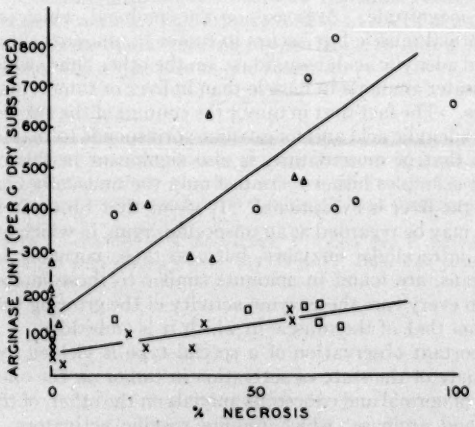


FIG. 7. Degree of activation of arginase and tumor aging.

× = Initial activity No. 1.
□ = Initial activity No. 256.
△ = Full activity No. 1.
○ = Full activity No. 256.

dipeptidase (see Table I of the Experimental Part*), like that of catalase, shows no significant dependence on the degree of aging of the tumor.

* To be published in the June issue of the JOURNAL.

The observed fluctuations may have to be explained by a varying infiltration of the tumors by leucocytes, as could be established by histological analyses in some of the cases observed.

In order to understand the changes of the enzymatic equipment in tumors and their significance, it is especially informative to follow the enzyme content of other organs in the animal. For some of the enzymes studied, as may be seen in the experimental part, significant changes are found in the amount of enzyme in the liver and kidneys of cancer animals. The content of cathepsin [2] and arginase in the liver is reduced, the phosphatase content of the kidneys increased. A profitable task which we have begun would be to follow stepwise the alteration of the organs not affected during the aging of the cancer, to arrive at a balance in the enzyme economics of cancerous animals. The changes of the enzyme composition of musculature, such as the appearance of arginase in cancerous animals as described by Klein and Ziese,[3] are pertinent to this question.

The comparison we have made of the enzyme content in tumors with that in the organs and the musculature of normal animals leads to the conclusion that in almost every case the enzyme content of tumor is similar to that of normal organs, but differs from that of muscle tissue in order of magnitude. Arginase, on the one hand, which is not found at all in normal muscle but occurs in tumor in amounts similar to that in liver, and adenylic acid desamidase, on the other hand, which is found in much greater amounts in muscle than in liver or tumor, may be taken as examples. The fact that in tumor the content of the two desamidases specific for adenylic acid and for guanine corresponds to that of the liver, but not to that of musculature, is also significant in this connection. Among the examples hitherto studied only the unusually high catalase content of the liver is exceptional. It seems that tumor in its enzyme equipment may be regarded as an unspecific organ, in which not only the ubiquitous intracellular enzymes, but also those normally localized in certain organs, are found in amounts similar to those in their specific organs. In every case the enzyme activity of the growing tumor differs entirely from that of the muscle in which it is embedded.

An important observation of a special type is yielded by the comparative study of the state of activation in tumor on the one hand and in the liver of normal and cancerous animals on the other, of the enzymes cathepsin and arginase, which require specific activators. The proportion of activated cathepsin in tumors independent of their age, is not appreciably different from that observed in liver (Fig. 6); this would indicate that a distinctive change in the quantity of cathepsin-activating

[2] Compare this with the observations of E. MASCHMANN AND E. HELMERT, Hoppe-Seyler's *Zeitschrift für physiologische Chemie*, 216: 163 (1933).

[3] B. G. KLEIN AND W. ZIESE, *Zeitschrift für Krebsforschung*, 37: 323 (1932).

sulfhydryl compounds does not occur in tumor tissue.[4] For arginase, however, the activation of which requires both a heavy metal and sulfhydryl, it has been shown that the amount of enzyme occurring in the activated state in tumors is not greatly increased during aging, while the total amount of arginase present, as measured by maximal cystein-iron activation, is at the same time increased many times[5] (see Fig. 7). Also in livers of cancerous animals the enzyme occurs in only partially activated form. The incompleteness of arginase activation is due, as the interpretation of the research shows, not so much to a lack of sulfhydryl as to a lack of heavy metal. Especially in cases with highly increased arginase content, in advanced necrosis, a striking, often a maximal, activation of the enzyme is attained by the addition of iron alone, while the activating effect of cystein is much less pronounced. This finding seems of great significance. The activation of the arginase is, then, as the researches shortly to be published show, bound up with an oxidation-reduction process in which sulfhydryl-bound heavy metal takes part. Activation of the enzyme by cystein-iron is observed only in the presence of oxygen; it does not occur in hydrogen atmosphere. These observations suggest the idea that the incompleteness of arginase activation in tumors is to be attributed to the same cause responsible for the decrease of normal respiration, possibly to a lack of an activating heavy metal with particular binding properties. To follow this concept farther is one of the objectives of our research.

(To be continued.)

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[4] Compare this with H. KLEINMANN AND F. WERR, *Biochemische Zeitschrift*, **241**: 108, 140, 181 (1931); P. RONDONI, *Biochemical Journal*, **26**: 1477 (1932); E. MASCHMANN AND E. HELMERT, Hoppe-Seyler's *Zeitschrift für physiologische Chemie*, **216**: 141, 171 (1933). Concerning the findings of the latter authors one of us will shortly enter into a discussion to be published elsewhere.

[5] Compare the observations of S. EDLBACHER, J. KRAUS AND F. LEUTHARDT, Hoppe-Seyler's *Zeitschrift für physiologische Chemie*, **217**: 89 (1933) which were published after the completion of our work.

CURRENT TOPICS.

Problems to Face in Testing Jet Engines. (*Heating and Ventilating*, Vol. 44, No. 10.)—Reconversion of an 18-year-old brick cell from its original use in testing conventional piston aircraft engines to testing jet-powered engines was recently completed by the Pratt and Whitney Division of United Aircraft. Principal problems encountered were:

(1) the heat of the exhaust, which reaches 1200 F.
(2) protection from flying particles from burst turbines or compressors, which travel with the velocity of machine gun projectiles.

These problems were studied jointly by engineers of Pratt and Whitney Aircraft Division and United Aircraft Corporation, and by Albert Kahn Associated Architects and Engineers, Inc., who are architects on this and other Pratt and Whitney expansion projects at East Hartford, Conn.

To meet the problem of heat, unsheathed brick walls were adjudged adequate in the exhaust flow at right angles and directed upward through an acoustically lined brick stack. At the point of turning, the 1200 F. at the orifice is reduced to approximately 500 F.

Plastic armor is used to meet the problem of protection from flying particles.

The cell is of brick construction, 12 ft. wide, 40 ft. long, and 20 ft. high. Two brick stacks project above the roof, one at each end, one for air intake and the other exhaust. None of these specifications in the original structure needed to be changed in the reconversion, except that the stacks were fitted with sliding covers to permit all-weather work inside.

Because the jet engine consists only of compressor, turbine and burner, no special new handling problems were entailed. Engines for test are brought in on an assembly stand equipped with wheels, and a one-ton monorail crane is used for service handling. Engines are moved to the test position through a removable section in the air entrance tube.

Plastic armor is used in the engine area to protect operators and control room. Steel plates are additionally used on the control room side. Two observation ports look out of the control room into the rear of the test area where engineers may examine the exhaust flame.

Some idea of the protective qualities required of the armor is the method of testing turbine rotor before assembly. These tests take place in pits where six inches of steel armor are backed by two feet of solid concrete.

R. H. O.

Electric Locomotives for Paulista Railway of Brazil.—Mass production methods are speeding the manufacture of 182-ton, 3,000-volt d-c electric passenger locomotives for the Paulista Railway of Brazil in the locomotive shops of the General Electric Company at Erie, Pa. Twelve of these locomotives have already been shipped.

Built for operation over mountainous terrain involving gradients as great as 1.85 per cent., these 3,000-volt d-c 2–C+C–2 locomotives are rated contin-

uously at 4,050-hp. with an hourly rating of 4,470-hp. With an over-all length of 76 feet, the weight on each of six driving axles is 45,000 pounds for a total of 270,000 pounds on driving axles.

The two-axle guiding trucks and two articulated three-axle driving trucks of these 2–C+C–2 locomotives are completely fabricated by welding. Considerable saving of time and labor is realized by this method in elimination of extensive machining and handling large heavy parts for machining.

Of all-steel, all-welded construction, the cab underframe is composed of two main 21-inch, I-beam longitudinal sills welded to cross sills and further braced and supported in position by heavy floor plates. The main air duct is a chamber formed in the space between the main longitudinal sills.

The cab sides and ends are built of separate assemblies, set in position on the underframe and then welded in place. The roof, consisting mainly of covered hatches, is constructed and assembled on the locomotive in a similar manner.

The high-voltage compartment, entirely enclosed and occupying the center section of the main apparatus cab, is designed to permit the complete installation of electrical equipment before the compartment itself is installed in the locomotive. After assembly inside the cab the complete unit is welded into place and final electrical connections are made. This compartment is equipped with removable side covers to give access to the equipment while end doors furnish entry to the compartment interior.

Designed for 1,500-volt service and insulated to operate two in series on 3,000-volt supply, the six traction motors are of the commutating-pole, force-ventilated type. Each motor is suspended from the axle by two constant-level oil filled waste-packed bearings and by a spring nose support carried on the truck transom. A single wide-faced pinion on the motor shaft engages a a gear of special heat-treated steel to complete the drive.

A three-speed system of control gives traction motor combinations of six in series; three in series, two groups in parallel; and two in series, three groups in parallel. Provision is also made for multiple-unit operation when such service is required.

R. H. O.

Pressure Gasometer Aids in Industrial Health Studies. (*Compressed Air*, Vol. 52, No. 10.)—An unusual application of compressed air has been found in the development of a pressure gasometer by the Bureau of Industrial Hygiene of the Detroit Department of Health. Before gas and vapor measuring devices can be used satisfactorily for the examination of workroom air, each instrument must be checked with a known concentration of the vapor to be determined. This is called "calibrating" the instrument and enables the bureau's engineers and chemists to make accurate determinations when estimating the amount of toxic vapors present in industrial atmospheres.

This type of calibration has always entailed considerable difficulties chiefly because it is not easy to keep constant amounts of solvent evaporating without gross fluctuations. Moreover an air chamber large enough to serve as an experimental room is not within the reach of most laboratories. The pressure gasometer, however, has solved the problem of constant amounts of vapor-air mixtures. The instrument itself consists of a sturdily constructed steel cham-

ber (air-compressor tank) with a capacity of about 4.7 cubic feet. It is provided with an inlet valve for solvent and air, a pressure gauge, a needle valve, and a safety valve.

Setting up and operating the apparatus is comparatively simple. The gasometer is first brought under reduced pressure by evacuating it. Then a weighed amount of the solvent whose vapor is to be tested is placed in a small flask and attached to the inlet valve. The latter is opened, and the inrushing air passes just above the surface of the solvent. By the time the chamber has reached atmospheric pressure, all the solvent has been swept in. Compressed air from a high-pressure cylinder is then admitted into the gasometer, where it mixes with the solvent vapor. When the desired amount of air has been let in, as registered by the gauge, the supply is shut off. A simple calculation then tells just what concentration of vapor and air exists in terms of parts per million. The maximum operating pressure of the gasometer is 150 pounds.

Opening the needle valve releases a stream of vapor-laden air, which is sufficient for a long series of calibrations. By starting off with a heavy concentration of solvent vapor, alternately calibrating instruments, and then admitting more air to dilute the residual moisture, it is easy to obtain a series of different known concentrations.

Only compressed air which is pure is used in the gasometer, and the bureau's laboratory personnel keeps close check on the manner in which it is supplied. This method of using compressed air has saved the bureau many hours of tedious work and has played an important role in assuring adequate health and safety factors in the Detroit workers' environment.

R. H. O.

Materials for Gas Turbine Service.—Great strides have been made in the development of suitable materials for gas turbine service within a comparatively limited time.

Selection of promising materials has been based on creep and rupture test results. Development of materials in use requires a knowledge of all properties of each material and extensive laboratory testing, including behavior under vibration or stress or temperature variation, heat shock, notch sensitivity, corrosion and erosion by hot gases, ductility and stability.

One difficulty which required a vast amount of testing to destruction was obtaining satisfactory large forgings of austenitic materials. This problem was best solved by the development of the so-called "composite" wheel. This consists of a central or hub and disk portion of ferritic low-temperature steel and an outer or rim portion of high-temperature austenitic material solidly bonded together by a suitably controlled weld, using austenitic weld rod. This design has an additional advantage, in that the hub forging can include the shaft extension, thus eliminating the necessity of welding on a shaft extension.

Despite the fact that much progress already has been made, extensive additional data are needed to develop new materials and to understand the laws governing high temperature material performance. A great deal must be done, too, to develop controls needed to ensure that materials always will possess the most desirable properties.

R. H. O.